On Graded Coalgebras of Graded Linear Exponential Comonad

Shin-ya Katsumata

Research Institute for Mathematical Sciences, Kyoto University

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The ! Modality of Linear Logic

... is the key structure to represent the non-linear world inside the linear world:

$$
\begin{array}{ll}
!A \vdash !!A & !A \vdash A \\
!A \otimes !B \vdash !(A \otimes B) & \mathbf{I} \vdash !\mathbf{I} \\
!A \vdash !A \otimes !A & !A \vdash \mathbf{I}
\end{array}
$$

$$A \Rightarrow B = !A \multimap B, \quad !(A \times B) \simeq !A \otimes !B$$

Linear Exponential Comonad

... consists of $! : \mathbb{C} \to \mathbb{C}$ and

$$
\begin{array}{ll}
\delta_X :!X \to !!X & \epsilon_X :!X \to X \\
m_{X,Y} :!X \otimes !Y \to !(X \otimes Y) & m_{()} : \mathbf{I} \to !\mathbf{I} \\
\gamma_X :!X \to !X \otimes !X & \omega_X :!X \to \mathbf{I}
\end{array}
$$

satisfying more than 20 equational axioms.

Theorem

Every symmetric lax monoidal adjunction of the following type yields a linear exponential comonad.

$$
(\mathbb{N}, 1, \times) \underset{R}{\overset{L}{\rightleftarrows}} (\mathbb{L}, \mathbf{I}, \otimes)
$$

Linear Exponential Comonad

Theorem (Resolution Theorem)

Let $!$ be a linear exponential comonad on a symmetric monoidal category $\mathbb{C}$. Then $\mathbb{C}^!$ has finite products, and the canonical adjunction is symmetric lax monoidal:

$$(\mathbb{C}^!, 1, \times) \underset{R}{\overset{L}{\rightleftarrows}} (\mathbb{C}, \mathbf{I}, \otimes)$$

Graded Linear Exponential Comonad

... annotates !-type with an element of a semiring R:

$$! : R \times \mathbb{C} \to \mathbb{C}$$

$$\delta_{r,s,X} : !_{r \times s} X \to !_r (!_s X) \qquad \epsilon_X : !_1 X \to X$$

$$m_{r,X,Y} : !_r X \otimes !_r Y \to !_r (X \otimes Y) \qquad m_{r,()} : \mathbf{I} \to !_r \mathbf{I}$$

$$\gamma_{r,s,X} : !_{r+s} X \to !_r X \otimes !_s X \qquad \omega_X : !_0 X \to \mathbf{I}$$

with more than 20 equational axioms.

- Bounded linear logic [Girard et al. '92]
- Coeffect calculus [Brunel et al. '14] (see also [Ghica et al. '14], [Petricek et al. '13], [Grellois and Melliès '15])
- A construction is given in [Breuvert and Pagani '15]

Motivation

We would like to extend the resolution theorem to the graded setting:

Theorem (Resolution Theorem, Revisited)

Let D be a linear exponential comonad on a symmetric monoidal category $\mathbb{C}$. Then $\mathbb{C}^D$ has finite products, and the canonical adjunction is symmetric lax monoidal:

$$(\mathbb{C}^D, 1, \times) \underset{R}{\overset{L}{\rightleftarrows}} (\mathbb{C}, \mathbf{I}, \otimes)$$

Motivation

Question

Let D be a graded linear exponential comonad on a symmetric monoidal category $\mathbb{C}$. Then

- How do we define $\mathbb{C}^D$ and the associated symmetric lax monoidal adjunction?

$$\mathbb{C}^D \underset{R}{\overset{L}{\rightleftarrows}} (\mathbb{C}, \mathbf{I}, \otimes)$$

- How can we recover D?

Contribution

Our Proposal

We introduce:

- the category $C(\mathbb{C}, D)$ of graded comonoid-coalgebras
- the twist functor $T : R \times C(\mathbb{C}, D) \to C(\mathbb{C}, D)$

$$C(\mathbb{C}, D) \underset{R}{\overset{L}{\rightleftarrows}} (\mathbb{C}, \mathbf{I}, \otimes)$$

such that $DrA = L \circ Tr \circ R(A)$.

c.f. For the case of graded monads, see [Fujii, K and Melliès '16]

Graded Linear Exponential Comonad

- $(R, \leq, 0, +, 1, \times)$: an ordered semiring

$$R^{+} \stackrel{\text{def}}{=} (R, \leq, 0, +), \quad R^{\times} \stackrel{\text{def}}{=} (R, \leq, 1, \times)$$

- $(\mathbb{C}, \mathbf{I}, \otimes)$: a SMC.
- $(\mathbf{SMon}_l[\mathbb{C}, \mathbb{C}], \dot{\mathbf{I}}, \dot{\otimes})$: the SMC where
  - Objects: symmetric lax monoidal endofunctors on $\mathbb{C}$
  - Morphisms: monoidal natural transformations
  - Symmetric monoidal structure:

$$\dot{\mathbf{I}}X = \mathbf{I}, \quad (F \mathbin{\dot{\otimes}} G)X = FX \otimes GX$$

Graded Linear Exponential Comonad

An *R-graded linear exponential comonad* consists of:

$$D : (R, \leq) \to \mathbf{SMon}_l[\mathbb{C}, \mathbb{C}]$$
$$(D, \omega, \gamma) : R^+ \to (\mathbf{SMon}_l[\mathbb{C}, \mathbb{C}], \dot{\mathbf{I}}, \dot{\otimes}) \quad \text{sym. colax}$$
$$(D, \epsilon, \delta) : R^\times \to (\mathbf{SMon}_l[\mathbb{C}, \mathbb{C}], \mathrm{Id}, \circ) \quad \text{colax}$$

satisfying...

Graded Linear Exponential Comonad

four extra axioms:

$$\begin{tikzcd} D(0s)A \arrow[d, "\delta"'] \arrow[dr, "\omega"] & \\ D0(DsA) \arrow[r, "\omega"'] & \mathbf{I} \end{tikzcd}$$

$$\begin{tikzcd} D((r+r')s)A \arrow[r, "\gamma"] \arrow[d, "\delta"'] & D(r \times s)A \otimes D(r' \times s)A \arrow[d, "\delta \otimes \delta"] \\ D(r+r')DsA \arrow[r, "\gamma"'] & Dr(DsA) \otimes Dr'(DsA) \end{tikzcd}$$

... corresponding to "δ is a comonoid morphism"

Graded Linear Exponential Comonad

four extra axioms:

$$
\begin{tikzcd}
D(s0)A \arrow[r, "\omega"] \arrow[d, "\delta"'] & \mathbf{I} \arrow[d, "m"] \\
Ds(D0A) \arrow[r, "Ds\omega"'] & Ds\mathbf{I}
\end{tikzcd}
$$

$$
\begin{tikzcd}
D(s(r+r'))A \arrow[r, "\gamma"] \arrow[d, "\delta"'] & D(sr)A \otimes D(sr')A \arrow[r, "\delta \otimes \delta"] & Ds(DrA) \otimes Ds(Dr'A) \arrow[d, "m"] \\
Ds(D(r+r')A) \arrow[rr, "Ds\gamma"'] & & Ds(DrA \otimes Dr'A)
\end{tikzcd}
$$

... corresponding to "ω, γ are coalgebra morphisms"

Graded Linear Exponential Comonad

In the category of metric spaces, metric scaling

$$Dr(X, d) = (X, r \times d)$$

is an $([0, \infty), \leq)$-graded linear exponential comonad.

In a symmetric monoidal category $\mathbb{C}$ with sufficient limits, the symmetric tensor product

$$DnX = \lim(X^{\otimes n} \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} X^{\otimes n})$$

is an $(\mathbf{N}, =)$-graded linear exponential comonad.

Graded Linear Exponential Comonad

Theorem

$D1$ is a symmetric lax monoidal comonad.

Theorem

Linear exponential comonads are exactly a 1-graded linear exponential comonads.

Motivation

Question

Let D be a graded linear exponential comonad on a symmetric monoidal category $\mathbb{C}$. Then

- How do we define $\mathbb{C}^D$ and the associated symmetric lax monoidal adjunction?

$$\mathbb{C}^D \underset{R}{\overset{L}{\rightleftarrows}} (\mathbb{C}, \mathbf{I}, \otimes)$$

Our proposal: Category of R-graded comonoid-coalgebras.

Graded Comonoid-Coalgebra

An R-graded comonoid-coalgebra (for D) consists of:

$A : (R, \leq) \to \mathbb{C}$

$a_{r,s} : A(rs) \to Dr(As)$ R-graded coalgebra

$(A, \omega, \gamma) : R^+ \to (\mathbb{C}, \mathbf{I}, \otimes)$ symmetric colax monoidal

$$\begin{tikzcd} Ar \arrow[r, "a"] \arrow[dr] & D1(Ar) \arrow[d, "\epsilon"] \\ & Ar \end{tikzcd} \qquad \begin{tikzcd} A(rst) \arrow[r, "a"] \arrow[d, "a"'] & D(rs)(At) \arrow[d, "\delta"] \\ Dr(A(st)) \arrow[r, "Dra"'] & Dr(Ds(At)) \end{tikzcd}$$

satisfying...

Graded Comonoid-Coalgebra

four extra axioms

$$\begin{tikzcd}
A(0 \times r) \arrow[d, "a"'] \arrow[dr, "w"] & \\
D0(Ar) \arrow[r, "\omega"'] & \mathbf{I}
\end{tikzcd}$$

$$\begin{tikzcd}
A((s+t) \times r) \arrow[r, "c"] \arrow[d, "a"'] & A(s \times r) \otimes A(t \times r) \arrow[d, "a \otimes a"] \\
D(s+t)(Ar) \arrow[r, "\gamma"'] & Ds(Ar) \otimes Dt(Ar)
\end{tikzcd}$$

Graded Comonoid-Coalgebra

four extra axioms

$$
\begin{tikzcd}
A0 \arrow[r, "w"] \arrow[d, "a"'] & \mathbf{I} \arrow[d, "m"] \\
Dr(A0) \arrow[r, "Drw"'] & Dr\mathbf{I}
\end{tikzcd}
$$

$$
\begin{tikzcd}
A(r \times s + r \times t) \arrow[r, "c"] \arrow[d, "a"'] & A(r \times s) \otimes A(r \times t) \arrow[r, "a \otimes a"] & Dr(As) \otimes Dr(At) \arrow[d, "m"] \\
Dr(A(s+t)) \arrow[rr, "Drc"'] & & Dr(As \otimes At)
\end{tikzcd}
$$

Graded Comonoid-Coalgebra

A morphism from (A, a, w, c) to (B, b, x, d) is a natural transformation $f : A \to B$ such that

- f is a graded coalgebra morphism:

$$\begin{tikzcd} A(rs) \arrow[r, "f_{rs}"] \arrow[d, "a_{r,s}"'] & B(rs) \arrow[d, "b_{r,s}"] \\ Dr(As) \arrow[r, "Dr(f_s)"'] & Dr(Bs) \end{tikzcd}$$

- f is a monoidal natural transformation from (A, w, c) to (B, x, d).

We form the category $C(\mathbb{C}, D)$ by these data.

Graded Comonoid-Coalgebra

Theorem

$C(\mathbb{C}, D)$ is a symmetric monoidal category.

$$(A, \cdots) \otimes (B, \cdots) = (\lambda r \ . \ Ar \otimes Br, \cdots)$$

There is an evident forgetful functor

$$C(\mathbb{C}, D) \longrightarrow \mathbb{C}^D$$

to the category of R-graded coalgebras for D.

Theorem

When $R = 1$, the above functor is an isomorphism.

Adjunction

$$
\begin{tikzcd}
C(\mathbb{C}, D) \arrow[d, shift right, "L"'] \\
\mathbb{C} \arrow[u, shift right, "R"']
\end{tikzcd}
$$

$$L(A, a, w, c) = A1$$

$$RX = (D - X, \delta_{-,=,X}, \omega_X, \gamma_{-,=,X})$$

Theorem

$L \dashv R$, L is strict monoidal and $L \circ R = D1$.

Question

How do we recover D itself?

Twist of Graded Comonoid-Coalgebras

For $r \in R$, we define an r-twist functor:

$$T(r) : C(\mathbb{C}, D) \to C(\mathbb{C}, D)$$

$$T(r)(A, a, w, c) \stackrel{\text{def}}{=} (A', a', w', c'), \quad T(r)(h)_s \stackrel{\text{def}}{=} h_{sr}$$

where

$$A's \stackrel{\text{def}}{=} A(sr), \quad a'_{s,t} \stackrel{\text{def}}{=} a_{s,tr}, \quad w' \stackrel{\text{def}}{=} w, \quad c'_{s,t} \stackrel{\text{def}}{=} c_{sr,tr}$$

Twist of Graded Comonoid-Coalgebras

Theorem

T is an R-graded linear exponential comonad on $C(\mathbb{C}, D)$ such that

- *For each $r \in R$, Tr is symmetric strict monoidal.*
- *$T1 = \mathrm{Id}$ and $T(r \times s) = Tr \circ Ts$.*

Corollary

When $R = 1$, $T* = \mathrm{Id}$ carries a commutative comonoid structure in $(\mathbf{SMon}_l[\mathbb{C}, \mathbb{C}], \dot{\mathbf{I}}, \dot{\otimes})$.

Theorem

$DrX = L \circ Tr \circ R(X)$.

What about Extra Four Axioms?

The definitions of

- graded linear exponential comonad
- graded comonoid-coalgebra

come with "extra four axioms".

Where are they from?

What about Extra Four Axioms?

The definitions of

- graded linear exponential comonad
- graded comonoid-coalgebra

come with "extra four axioms".

Where are they from? $\Rightarrow$ A double category helps.

Double Category of SMCs

... in [Grandis&Pare 2004] consists of:

- 0-cell: symmetric monoidal category
- horizontal 1-cell: symmetric lax monoidal functor
- vertical 1-cell: symmetric colax monoidal functor
- 2-cell:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "H"] \arrow[d, "V'"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathbb{D} \arrow[d, "V"] \\ \mathbb{E} \arrow[r, "H'"'] & \mathbb{F} \end{tikzcd}$$

satisfying...

Double Category of Monoidal Categories

two equations

$$\begin{tikzcd} V\mathbf{I} \arrow[r] \arrow[d] & V(H\mathbf{I}) \arrow[r, "\alpha"] & H'(V'\mathbf{I}) \arrow[d] \\ \mathbf{I} \arrow[rr] & & H'\mathbf{I} \end{tikzcd}$$

$$\begin{tikzcd} V(HX \otimes HY) \arrow[r] \arrow[d] & V(H(X \otimes Y)) \arrow[r, "\alpha"] & H'(V'(X \otimes Y)) \arrow[d] \\ V(HX) \otimes V(HY) \arrow[r, "\alpha \otimes \alpha"'] & H'(V'X) \otimes H'(V'Y) \arrow[r] & H'(V'X \otimes V'Y) \end{tikzcd}$$

This reduces to the axiom of monoidal natural transformation when $V = V' = \mathrm{Id}$ or $H = H' = \mathrm{Id}$.

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$$(D, \epsilon, \delta) : R^\times \to (\mathbf{SMon}_l[\mathbb{C}, \mathbb{C}], \mathrm{Id}, \circ) \quad \text{colax}$$

Let us view δ as:

$$\delta : D(r \times r') \to Dr \circ Dr' : R \times R \to \mathbf{SMon}_l[\mathbb{C}, \mathbb{C}]$$

This should satisfy...

Graded Linear Exponential Comonad

For each $r \in R^+$,

$$\begin{tikzcd} R^+ \arrow[rr, "- \times r"] \arrow[d, "D"'] & \arrow[d, phantom, "\Downarrow \delta_{-,r}"] & R^+ \arrow[d, "D"] \\ \mathbf{SMon}_l[\mathbb{C},\mathbb{C}] \arrow[rr, "- \circ Dr"'] & & \mathbf{SMon}_l[\mathbb{C},\mathbb{C}] \end{tikzcd}$$

$$\begin{tikzcd} R^+ \arrow[rr, "r \times -"] \arrow[d, "D"'] & \arrow[d, phantom, "\Downarrow \delta_{r,-}"] & R^+ \arrow[d, "D"] \\ \mathbf{SMon}_l[\mathbb{C},\mathbb{C}] \arrow[rr, "Dr \circ -"'] & & \mathbf{SMon}_l[\mathbb{C},\mathbb{C}] \end{tikzcd}$$

are 2-cells of the double category of symmetric monoidal categories.

Graded Comonoid-Coalgebra

An *R-graded comonoid coalgebra* consists of:

$A : (R, \leq) \to \mathbb{C}$
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$(A, \omega, \gamma) : R^+ \to (\mathbb{C}, \mathbf{I}, \otimes)$ symmetric colax monoidal

such that...

Graded Linear Exponential Comonad

For each $r \in R^+$,

$$
\begin{tikzcd}
R^+ \arrow[r, "- \times r"] \arrow[d, "D"'] \arrow[dr, phantom, "\Downarrow a_{-,r}"] & R^+ \arrow[d, "A"] \\
\mathbf{SMon}_l[\mathbb{C}, \mathbb{C}] \arrow[r, "-(Ar)"'] & \mathbb{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
R^+ \arrow[r, "r \times -"] \arrow[d, "A"'] \arrow[dr, phantom, "\Downarrow a_{r,-}"] & R^+ \arrow[d, "A"] \\
\mathbb{C} \arrow[r, "Dr"'] & \mathbb{C}
\end{tikzcd}
$$

are 2-cells of the double category of symmetric monoidal categories.

Conclusion

- A resolution of graded linear exponential comonad.
- An alternative understanding of “extra four axioms” in terms of the double category of symmetric monoidal categories by Grandis and Pare.

Future Work:

- Setting up the category of resolutions of graded linear exponential comonads
- The finality of $C(\mathbb{C}, D)$
- What about the co-Kleisli category?