

On Graded Coalgebras of Graded Linear Exponential Comonad

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The ! Modality of Linear Logic

... is the key structure to represent the non-linear world inside the linear world:

$$!A \vdash !!A$$

$$!A \vdash A$$

$$!A \otimes !B \vdash !(A \otimes B)$$

$$\mathbf{I} \vdash !\mathbf{I}$$

$$!A \vdash !A \otimes !A$$

$$!A \vdash \mathbf{I}$$

$$A \Rightarrow B = !A \multimap B, \quad !(A \times B) \simeq !A \otimes !B$$

Linear Exponential Comonad

... consists of $! : \mathbb{C} \rightarrow \mathbb{C}$ and

$$\delta_X : !X \rightarrow !!X$$

$$\epsilon_X : !X \rightarrow X$$

$$m_{X,Y} : !X \otimes !Y \rightarrow !(X \otimes Y)$$

$$m_{\mathbb{I}} : \mathbb{I} \rightarrow !\mathbb{I}$$

$$\gamma_X : !X \rightarrow !X \otimes !X$$

$$\omega_X : !X \rightarrow \mathbb{I}$$

satisfying more than 20 equational axioms.

Theorem

Every symmetric lax monoidal adjunction of the following type yields a linear exponential comonad.

$$(\mathbb{N}, \mathbb{1}, \times) \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} (\mathbb{L}, \mathbb{I}, \otimes)$$

Linear Exponential Comonad

Theorem (Resolution Theorem)

Let $!$ be a linear exponential comonad on a symmetric monoidal category \mathbb{C} . Then $\mathbb{C}^!$ has finite products, and the canonical adjunction is symmetric lax monoidal:

$$(\mathbb{C}^!, \mathbf{1}, \times) \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} (\mathbb{C}, \mathbf{I}, \otimes)$$

Graded Linear Exponential Comonad

... annotates $!$ -type with an element of a semiring R :

$$! : R \times \mathbb{C} \rightarrow \mathbb{C}$$

$$\delta_{r,s,X} : !_{r \times s} X \rightarrow !_r (!_s X)$$

$$\epsilon_X : !_1 X \rightarrow X$$

$$m_{r,X,Y} : !_r X \otimes !_r Y \rightarrow !_r (X \otimes Y)$$

$$m_{r,()} : \mathbf{!} \rightarrow !_r \mathbf{!}$$

$$\gamma_{r,s,X} : !_{r+s} X \rightarrow !_r X \otimes !_s X$$

$$\omega_X : !_0 X \rightarrow \mathbf{!}$$

with more than 20 equational axioms.

- Bounded linear logic [Girard et al. '92]
- Coeffect calculus [Brunel et al. '14] (see also [Ghica et al. '14], [Petricek et al. '13], [Grellois and Melliès '15])
- A construction is given in [Breuvert and Pagani '15]

Motivation

We would like to extend the resolution theorem to the graded setting:

Theorem (Resolution Theorem, Revisited)

Let D be a linear exponential comonad on a symmetric monoidal category \mathbb{C} . Then \mathbb{C}^D has finite products, and the canonical adjunction is symmetric lax monoidal:

$$(\mathbb{C}^D, \mathbf{1}, \times) \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} (\mathbb{C}, \mathbf{I}, \otimes)$$

Question

Let D be a **graded** linear exponential comonad on a symmetric monoidal category \mathbb{C} . Then

- How do we define \mathbb{C}^D and the associated symmetric lax monoidal adjunction?

$$\mathbb{C}^D \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} (\mathbb{C}, \mathbf{I}, \otimes)$$

- How can we recover D ?

Our Proposal

We introduce:

- the category $C(\mathbb{C}, D)$ of **graded comonoid-coalgebras**
- the **twist functor** $T : R \times C(\mathbb{C}, D) \rightarrow C(\mathbb{C}, D)$

$$C(\mathbb{C}, D) \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} (\mathbb{C}, \mathbf{I}, \otimes)$$

such that $DrA = L \circ Tr \circ R(A)$.

c.f. For the case of graded monads, see [Fujii, K and Mellès '16]

Graded Linear Exponential Comonad

- $(R, \leq, 0, +, 1, \times)$: an ordered semiring

$$R^+ \stackrel{\text{def}}{=} (R, \leq, 0, +), \quad R^\times \stackrel{\text{def}}{=} (R, \leq, 1, \times)$$

- $(\mathbb{C}, \mathbf{I}, \otimes)$: a SMC.
- $(\mathbf{SMon}_l[\mathbb{C}, \mathbb{C}], \dot{\mathbf{I}}, \dot{\otimes})$: the SMC where
 - ▶ Objects: symmetric lax monoidal endofunctors on \mathbb{C}
 - ▶ Morphisms: monoidal natural transformations
 - ▶ Symmetric monoidal structure:

$$\dot{\mathbf{I}}X = \mathbf{I}, \quad (F \dot{\otimes} G)X = FX \otimes GX$$

Graded Linear Exponential Comonad

An R -graded linear exponential comonad consists of:

$$D : (R, \leq) \rightarrow \mathbf{SMon}_I[\mathbb{C}, \mathbb{C}]$$

$$(D, \omega, \gamma) : R^+ \rightarrow (\mathbf{SMon}_I[\mathbb{C}, \mathbb{C}], \mathbf{i}, \dot{\otimes}) \quad \text{sym. colax}$$

$$(D, \epsilon, \delta) : R^\times \rightarrow (\mathbf{SMon}_I[\mathbb{C}, \mathbb{C}], \text{Id}, \circ) \quad \text{colax}$$

satisfying...

Graded Linear Exponential Comonad

four extra axioms:

$$\begin{array}{ccc} D(0s)A & & \\ \delta \downarrow & \searrow \omega & \\ D0(DsA) & \xrightarrow{\omega} & \mathbf{I} \end{array}$$

$$\begin{array}{ccc} D((r + r')s)A & \xrightarrow{\gamma} & D(r \times s)A \otimes D(r' \times s)A \\ \delta \downarrow & & \downarrow \delta \otimes \delta \\ D(r + r')DsA & \xrightarrow{\gamma} & Dr(DsA) \otimes Dr'(DsA) \end{array}$$

... corresponding to “ δ is a comonoid morphism”

Graded Linear Exponential Comonad

four extra axioms:

$$\begin{array}{ccc}
 D(s0)A & \xrightarrow{\omega} & \mathbf{1} \\
 \delta \downarrow & & \downarrow m \\
 Ds(D0A) & \xrightarrow{Ds\omega} & Ds\mathbf{1}
 \end{array}$$

$$\begin{array}{ccccc}
 D(s(r + r'))A & \xrightarrow{\gamma} & D(sr)A \otimes D(sr')A & \xrightarrow{\delta \otimes \delta} & Ds(DrA) \otimes Ds(Dr'A) \\
 \delta \downarrow & & & & \downarrow m \\
 Ds(D(r + r')A) & \xrightarrow{Ds\gamma} & & & Ds(DrA \otimes Dr'A)
 \end{array}$$

... corresponding to “ ω, γ are coalgebra morphisms”

Graded Linear Exponential Comonad

In the category of metric spaces, metric scaling

$$Dr(X, d) = (X, r \times d)$$

is an $([0, \infty), \leq)$ -graded linear exponential comonad.

In a symmetric monoidal category \mathbb{C} with sufficient limits, the symmetric tensor product

$$DnX = \lim(X^{\otimes n} \begin{array}{c} \xrightarrow{\quad} \\ \vdots \\ \xrightarrow{\quad} \end{array} X^{\otimes n})$$

is an $(\mathbf{N}, =)$ -graded linear exponential comonad.

Graded Linear Exponential Comonad

Theorem

$D1$ is a symmetric lax monoidal comonad.

Theorem

Linear exponential comonads are exactly a 1-graded linear exponential comonads.

Motivation

Question

Let D be a **graded** linear exponential comonad on a symmetric monoidal category \mathbb{C} . Then

- How do we define \mathbb{C}^D and the associated symmetric lax monoidal adjunction?

$$\mathbb{C}^D \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} (\mathbb{C}, \mathbf{I}, \otimes)$$

Our proposal: Category of **R -graded comonoid-coalgebras**.

Graded Comonoid-Coalgebra

An R -graded comonoid-coalgebra (for D) consists of:

$$A : (R, \leq) \rightarrow \mathbb{C}$$

$$a_{r,s} : A(rs) \rightarrow Dr(As) \quad R\text{-graded coalgebra}$$

$$(A, \omega, \gamma) : R^+ \rightarrow (\mathbb{C}, \mathbf{I}, \otimes) \quad \text{symmetric colax monoidal}$$

$$\begin{array}{ccc}
 Ar & \xrightarrow{a} & D1(Ar) \\
 & \searrow & \downarrow \epsilon \\
 & & Ar \\
 \\
 A(rst) & \xrightarrow{a} & D(rs)(At) \\
 a \downarrow & & \downarrow \delta \\
 Dr(A(st)) & \xrightarrow{Dra} & Dr(Ds(At))
 \end{array}$$

satisfying...

Graded Comonoid-Coalgebra

four extra axioms

$$\begin{array}{ccc} A(0 \times r) & & \\ a \downarrow & \searrow w & \\ D0(Ar) & \xrightarrow{\omega} & \mathbf{I} \end{array}$$

$$\begin{array}{ccc} A((s+t) \times r) & \xrightarrow{c} & A(s \times r) \otimes A(t \times r) \\ a \downarrow & & \downarrow a \otimes a \\ D(s+t)(Ar) & \xrightarrow{\gamma} & Ds(Ar) \otimes Dt(Ar) \end{array}$$

Graded Comonoid-Coalgebra

four extra axioms

$$\begin{array}{ccc} A0 & \xrightarrow{w} & \mathbf{1} \\ a \downarrow & & \downarrow m \\ Dr(A0) & \xrightarrow{Drw} & Dr\mathbf{1} \end{array}$$

$$\begin{array}{ccc} A(r \times s + r \times t) & \xrightarrow{c} & A(r \times s) \otimes A(r \times t) \xrightarrow{a \otimes a} Dr(As) \otimes Dr(At) \\ a \downarrow & & \downarrow m \\ Dr(A(s + t)) & \xrightarrow{Drc} & Dr(As \otimes At) \end{array}$$

Graded Comonoid-Coalgebra

A morphism from (A, a, w, c) to (B, b, x, d) is a natural transformation $f : A \rightarrow B$ such that

- f is a graded coalgebra morphism:

$$\begin{array}{ccc} A(rs) & \xrightarrow{f_{rs}} & B(rs) \\ a_{r,s} \downarrow & & \downarrow b_{r,s} \\ Dr(As) & \xrightarrow{Dr(f_s)} & Dr(Bs) \end{array}$$

- f is a monoidal natural transformation from (A, w, c) to (B, x, d) .

We form the category $C(\mathbb{C}, D)$ by these data.

Graded Comonoid-Coalgebra

Theorem

$C(\mathbb{C}, D)$ is a symmetric monoidal category.

$$(A, \dots) \otimes (B, \dots) = (\lambda r . Ar \otimes Br, \dots)$$

There is an evident forgetful functor

$$C(\mathbb{C}, D) \longrightarrow \mathbb{C}^D$$

to the category of R -graded coalgebras for D .

Theorem

When $R = 1$, the above functor is an isomorphism.

Adjunction

$$\begin{array}{c} C(\mathbb{C}, D) \\ L \downarrow \uparrow R \\ \mathbb{C} \end{array}$$

$$L(A, a, w, c) = A1$$

$$RX = (D - X, \delta_{-,=,X}, \omega_X, \gamma_{-,=,X})$$

Theorem

$L \dashv R$, L is strict monoidal and $L \circ R = D1$.

Question

How do we recover D itself?

Twist of Graded Comonoid-Coalgebras

For $r \in R$, we define an r -twist functor:

$$T(r) : C(\mathbb{C}, D) \rightarrow C(\mathbb{C}, D)$$

$$T(r)(A, a, w, c) \stackrel{\text{def}}{=} (A', a', w', c'), \quad T(r)(h)_s \stackrel{\text{def}}{=} h_{sr}$$

where

$$A's \stackrel{\text{def}}{=} A(sr), \quad a'_{s,t} \stackrel{\text{def}}{=} a_{s,tr}, \quad w' \stackrel{\text{def}}{=} w, \quad c'_{s,t} \stackrel{\text{def}}{=} c_{sr,tr}$$

Twist of Graded Comonoid-Coalgebras

Theorem

T is an R -graded linear exponential comonad on $C(\mathbb{C}, D)$ such that

- For each $r \in R$, Tr is symmetric *strict* monoidal.
- $T1 = \text{Id}$ and $T(r \times s) = Tr \circ Ts$.

Corollary

When $R = 1$, $T^* = \text{Id}$ carries a commutative comonoid structure in $(\mathbf{SMon}_I[\mathbb{C}, \mathbb{C}], \mathbf{i}, \dot{\otimes})$.

Theorem

$$DrX = L \circ Tr \circ R(X).$$

What about Extra Four Axioms?

The definitions of

- graded linear exponential comonad
- graded comonoid-coalgebra

come with “extra four axioms”.

Where are they from?

What about Extra Four Axioms?

The definitions of

- graded linear exponential comonad
- graded comonoid-coalgebra

come with “extra four axioms”.

Where are they from? \Rightarrow A double category helps.

Double Category of SMCs

... in [Grandis&Pare 2004] consists of:

- 0-cell: symmetric monoidal category
- horizontal 1-cell: symmetric lax monoidal functor
- vertical 1-cell: symmetric colax monoidal functor
- 2-cell:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{H} & \mathbb{D} \\ V' \downarrow & \Downarrow \alpha & \downarrow V \\ \mathbb{E} & \xrightarrow{H'} & \mathbb{F} \end{array}$$

satisfying...

Double Category of Monoidal Categories

two equations

$$\begin{array}{ccccc}
 V\mathbf{I} & \longrightarrow & V(H\mathbf{I}) & \xrightarrow{\alpha} & H'(V'\mathbf{I}) \\
 \downarrow & & & & \downarrow \\
 \mathbf{I} & \longrightarrow & & & H'\mathbf{I}
 \end{array}$$

$$\begin{array}{ccccccc}
 V(HX \otimes HY) & \longrightarrow & V(H(X \otimes Y)) & \xrightarrow{\alpha} & H'(V'(X \otimes Y)) \\
 \downarrow & & & & \downarrow \\
 V(HX) \otimes V(HY) & \xrightarrow{\alpha \otimes \alpha} & H'(V'X) \otimes H'(V'Y) & \longrightarrow & H'(V'X \otimes V'Y)
 \end{array}$$

This reduces to the axiom of monoidal natural transformation when $V = V' = \text{Id}$ or $H = H' = \text{Id}$.

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$$(D, \omega, \gamma) : R^+ \rightarrow (\mathbf{SMon}_I[\mathbb{C}, \mathbb{C}], \mathbf{i}, \dot{\otimes}) \quad \text{sym. colax}$$

$$(D, \epsilon, \delta) : R^\times \rightarrow (\mathbf{SMon}_I[\mathbb{C}, \mathbb{C}], \text{Id}, \circ) \quad \text{colax}$$

Let us view δ as:

$$\delta : D(r \times r') \rightarrow Dr \circ Dr' : R \times R \rightarrow \mathbf{SMon}_I[\mathbb{C}, \mathbb{C}]$$

This should satisfy...

Graded Linear Exponential Comonad

For each $r \in R^+$,

$$\begin{array}{ccc}
 R^+ & \xrightarrow{- \times r} & R^+ \\
 D \downarrow & \Downarrow \delta_{-,r} & \downarrow D \\
 \mathbf{SMon}_I[\mathbb{C}, \mathbb{C}] & \xrightarrow{- \circ Dr} & \mathbf{SMon}_I[\mathbb{C}, \mathbb{C}]
 \end{array}$$

$$\begin{array}{ccc}
 R^+ & \xrightarrow{r \times -} & R^+ \\
 D \downarrow & \Downarrow \delta_{r,-} & \downarrow D \\
 \mathbf{SMon}_I[\mathbb{C}, \mathbb{C}] & \xrightarrow{Dr \circ -} & \mathbf{SMon}_I[\mathbb{C}, \mathbb{C}]
 \end{array}$$

are 2-cells of the double category of symmetric monoidal categories.

Graded Comonoid-Coalgebra

An R -graded comonoid coalgebra consists of:

$$A : (R, \leq) \rightarrow \mathbb{C}$$

$$a_{r,s} : A(rs) \rightarrow Dr(As) : R \times R \rightarrow \mathbb{C} \quad R\text{-graded coalgebra}$$

$$(A, \omega, \gamma) : R^+ \rightarrow (\mathbb{C}, \mathbf{1}, \otimes) \quad \text{symmetric colax monoidal}$$

such that...

Graded Linear Exponential Comonad

For each $r \in R^+$,

$$\begin{array}{ccc}
 R^+ & \xrightarrow{- \times r} & R^+ \\
 D \downarrow & \Downarrow a_{-,r} & \downarrow A \\
 \mathbf{SMon}_l[\mathbb{C}, \mathbb{C}] & \xrightarrow{-(Ar)} & \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 R^+ & \xrightarrow{r \times -} & R^+ \\
 A \downarrow & \Downarrow a_{r,-} & \downarrow A \\
 \mathbb{C} & \xrightarrow{Dr} & \mathbb{C}
 \end{array}$$

are 2-cells of the double category of symmetric monoidal categories.

Conclusion

- A resolution of graded linear exponential comonad.
- An alternative understanding of “extra four axioms” in terms of the double category of symmetric monoidal categories by Grandis and Pare.

Future Work:

- Setting up the category of resolutions of graded linear exponential comonads
- The finality of $C(\mathbb{C}, D)$
- What about the co-Kleisli category?