A fibrational account of local states and beyond

Paul-André Melliès

Institut de Recherche en Informatique Fondamentale (IRIF)
CNRS & Université Paris Diderot

Crecogi Meeting
Université Aix * Marseille
Sunday 28 August 2016
The local state monad

A monad with graded arities
Presheaf models

**Key idea:** interpret a type $A$ as a family of sets

$$A_0 \quad A_1 \quad \cdots \quad A_n \quad \cdots$$

indexed by natural numbers, where each set $A_n$ contains the programs of type $A$ which have access to $n$ variables.
Presheaf models

This defines a covariant presheaf

$$A_n : \text{Inj} \rightarrow \text{Set}$$
on the category $\text{Inj}$ of natural numbers and injections.

Every injection $f : p \rightarrow q$ induces a function

$$A_f : A_p \rightarrow A_q$$

obtained by renaming every register $i \in [p]$ by the register $f(i) \in [q]$. 
Local stores [Plotkin & Power 2002]

The slightly intimidating monad

\[
LA : n \mapsto S^n \Rightarrow \left( \int_{p \in \text{Inj}} S^p \times A_p \times \text{Inj}(n,p) \right)
\]

on the presheaf category \([\text{Inj}, \text{Set}]\) where the contravariant presheaf

\[
S^p = \text{Val}^p
\]

describes the states available at degree \(p\).
Graded arities

A graded arity is defined as a finite sum of representables

\[
[ p_0, \cdots, p_n ] = \langle 0 \rangle + \cdots + \langle 0 \rangle + \cdots + \langle n \rangle + \cdots + \langle n \rangle
\]

\[ p_0 \text{ times} \]
\[ p_n \text{ times} \]

where each representable presheaf

\[ \langle n \rangle : \text{Inj} \rightarrow \text{Set} \]

is defined as

\[ \langle n \rangle = p \mapsto \text{Inj}(n, p) \]
Graded arities

This defines a full and faithful functor

\[ \Sigma \text{Inj}^{op} \rightarrow \text{Inj}^{op} \]

which generalizes the full and faithful functor

\[ \text{FinSet} \rightarrow \text{Set} \]

encountered in the case of finitary monads.
Monads with graded arities

Definition. A monad

\[ T : \text{[Inj, Set]} \rightarrow \text{[Inj, Set]} \]

has graded arities when the diagram

exhibits the functor \( T \) as a left Kan extension of \( T \circ i \) along \( i \).
The local state monad

**Theorem [RTA-TLCA 2014]** The local state monad

\[
L_A : n \mapsto S^n \Rightarrow \left( \int_{p \in \text{Inj}} S^p \times A_p \times \text{Inj}(n,p) \right)
\]

is a monad with **graded arities**.
The local state monad

For that reason, the local state monad

$$L : [\text{Inj, Set}] \rightarrow [\text{Inj, Set}]$$

is entirely described by its restriction

$$L \circ i : \Sigma \text{Inj}^{\text{op}} \xrightarrow{i} [\text{Inj, Set}] \xrightarrow{L} [\text{Inj, Set}]$$

to the subcategory $\Sigma \text{Inj}^{\text{op}}$ of graded arities.
Key observation behind this result

The local state monad

\[ L : [\text{Inj, Set}] \rightarrow [\text{Inj, Set}] \]

is the composite

\[ L = \mathcal{F} \circ \mathcal{B} \]

of two monads with graded arities

\[ \mathcal{F}, \mathcal{B} : [\text{Inj, Set}] \rightarrow [\text{Inj, Set}] \]

related by a distributivity law

\[ \lambda : \mathcal{B} \circ \mathcal{F} \Rightarrow \mathcal{F} \circ \mathcal{B}. \]
The fiber monad and the basis monad

The fiber monad

\[ \mathcal{F} A : n \mapsto S^n \Rightarrow (S^n \times A_n) \]

is the **global state monad** applied on each fiber \( n \) while

\[ \mathcal{B} = \ell^* \circ \exists \ell : [\text{Inj}, \text{Set}] \longrightarrow [\text{Inj}, \text{Set}] \]

is the **change-of-basis monad** associated to a functor

\[ \ell : \text{Inj} \longrightarrow \text{Res}. \]
An even nicer formulation

The functor \( \text{Inj} \xrightarrow{\ell} \text{Res} \) induces an adjunction:

\[
\begin{array}{ccc}
\exists_\ell & \downarrow & \exists_\ell \\
[\text{Inj}, \text{Set}] & \perp & [\text{Res}, \text{Set}] \\
\ell^* & \uparrow & \ell^*
\end{array}
\]

and the fiber monad lifts to a monad

\[ \mathcal{F} : [\text{Res}, \text{Set}] \longrightarrow [\text{Res}, \text{Set}] \]

Factorisation theorem [RTA-TLCA 2014]

The local state monad \( L \) coincides with the monad \( \ell^* \circ \mathcal{F} \circ \exists_\ell \).
What does this factorisation tell us?

Lookup ← Update ← Discharge ← Allocate ← Permute
Local mnemoids

Definition. A local mnemoid is a family of sets

\[ A_0 \quad A_1 \quad \cdots \quad A_n \quad \cdots \]

equipped with the following operations

\[
\text{lookup}_{\langle \text{loc} \rangle} : A_n \times A_n \rightarrow A_n
\]

\[
\text{update}_{\langle \text{loc, val} \rangle} : A_n \rightarrow A_n
\]

\[
\text{fresh}_{\langle \text{loc, val} \rangle} : A_{n+1} \rightarrow A_n
\]

\[
\text{dispose}_{\langle \text{loc} \rangle} : A_n \rightarrow A_{n+1}
\]

\[
\text{permute}_{\langle \text{loc} \rangle} : A_n \rightarrow A_n
\]

satisfying a series of elementary equations.
An algebraic presentation of the fiber monad
Creation lookup – update
Interaction update – update

\[ val_2 \cdot val_1 = val_2 \]
Interaction update – lookup
Commutation update – update
An algebraic presentation of the basis monad
Interaction fresh – update

\[ \text{val}_2 \begin{array}{c} \text{val}_1 \end{array} = \text{val}_2 \]
Commutation fresh – update

\[
\begin{array}{c}
\begin{array}{c}
\text{val}_1 \quad \text{val}_2 \\
\end{array} \\
\begin{array}{c}
D \\
D \\
D \\
D \\
\end{array} \\
\begin{array}{c}
D \\
D \\
D \\
D \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\text{val}_2 \\
\text{val}_1 \\
\end{array} \\
\begin{array}{c}
D \\
D \\
D \\
D \\
\end{array} \\
\begin{array}{c}
D \\
D \\
D \\
D \\
\end{array}
\end{array}
\]
Commutation fresh – update

\[ x \quad D \quad D \quad D \quad x \quad y \quad val \]

\[ = \]

\[ x \quad val \quad x \quad val \quad y \quad D \quad D \quad D \quad D \quad D \]
Interaction collect – update
Interaction permute – lookup

\[
\begin{array}{c}
D \\
D \\
D \\
D \\
\hline
D \\
\hline
D \\
\hline
D \\
\hline
D \\
\end{array}
\begin{array}{c}
D \\
D \\
D \\
D \\
\hline
D \\
\hline
D \\
\hline
D \\
\hline
D \\
\end{array}
= \\
\begin{array}{c}
D \\
D \\
D \\
D \\
\hline
D \\
\hline
D \\
\hline
D \\
\hline
D \\
\end{array}
\begin{array}{c}
D \\
D \\
D \\
D \\
\hline
D \\
\hline
D \\
\hline
D \\
\hline
D \\
\end{array}
\]
Local mnemoids

Definition. A local mnemoid is a family of sets

\[ A_0 \quad A_1 \quad \cdots \quad A_n \quad \cdots \]

equipped with the following operations

\[
\begin{align*}
\text{lookup}_{\langle \text{loc} \rangle} & : \quad A_n \times A_n \quad \rightarrow \quad A_n \\
\text{update}_{\langle \text{loc, val} \rangle} & : \quad A_n \quad \rightarrow \quad A_n \\
\text{fresh}_{\langle \text{loc, val} \rangle} & : \quad A_{n+1} \quad \rightarrow \quad A_n \\
\text{dispose}_{\langle \text{loc} \rangle} & : \quad A_n \quad \rightarrow \quad A_{n+1} \\
\text{permute}_{\langle \text{loc} \rangle} & : \quad A_n \quad \rightarrow \quad A_n
\end{align*}
\]

satisfying the equations above.
An algebraic presentation of the local state monad
[RTA-TLCA 2014]

the category of local mnemoids

is equivalent to

the category of algebras of the local state monad

Lookup ← Update ← Discharge ← Allocate ← Permute
The Grothendieck construction

A fibrational way to glue effects
The Grothendieck construction

An indexed \( \mathcal{B} \)-category is a pseudo functor from a category \( \mathcal{B} \)

\[
\mathcal{P} : \mathcal{B}^{\text{op}} \longrightarrow \text{Cat}
\]

which associates to every \( b \in \mathcal{B} \) a category \( \mathcal{P}(b) \).

The Grothendieck construction “glues” together the \( \mathcal{P}(b) \)’s into a fibration

\[
p : E \longrightarrow \mathcal{B}
\]

whose fiber

\[
E_b = p^{-1}(b)
\]

above the object \( b \in \mathcal{B} \) coincides with the category \( \mathcal{P}(b) \).
A morphism $f : a' \to a$ in $\mathcal{E}$ is cartesian above $u : b' \to b$ in $\mathcal{B}$ when the following property holds:

- for every morphism $g : a'' \to a$ for every morphism $v : b'' \to b'$ such that $u \circ v = p(g)$
- there exists a unique morphism $h : a'' \to a'$ such that $f \circ h = g$ and $p(h) = v$. 

\[
\begin{array}{c}
\text{Cartesian morphisms} \\
\text{A morphism } f : a' \to a \text{ in } \mathcal{E} \text{ is cartesian above } u : b' \to b \text{ in } \mathcal{B} \text{ when the following property holds:} \\
\text{for every morphism } g : a'' \to a \text{ for every morphism } v : b'' \to b' \text{ such that } u \circ v = p(g) \text{ there exists a unique morphism } h : a'' \to a' \text{ such that } f \circ h = g \text{ and } p(h) = v.
\end{array}
\]
Fibrations

Definition. A fibration is a functor

\[ p : \mathcal{E} \rightarrow \mathcal{B} \]

when there exists a cartesian morphism

\[
\begin{array}{ccc}
  u^*a & \xrightarrow{f} & a \\
    & \downarrow{p} & \\
  b' & \xrightarrow{u} & b
\end{array}
\]

for every object \( a \) in the fiber of \( b \) and every morphism \( u : b' \rightarrow b \).
The Grothendieck construction

The construction can be adapted to a pseudo 2-functor

\[ \mathcal{P} : \mathcal{B}^{op(1,2)} \to \text{Cat} \]

from a 2-category $\mathcal{B}$.

One obtains in that case a 2-functor

\[ p : \mathcal{E} \to \mathcal{B} \]

satisfying the following fibrational properties.
Fibrations: the 2-categorical case

Extended definition. A fibration is a 2-functor

\[ p : \mathcal{E} \longrightarrow \mathcal{B} \]

satisfying the following properties:

1. the underlying functor is a fibration (in the usual sense)

2. the functor

\[ p(a,a') : \mathcal{E}(a,a') \longrightarrow \mathcal{B}(p(a), p(a')) \]

is a discrete fibration for every \( a, a' \in \mathcal{E} \).
Cartesian 2-cells

A 2-cell $\varphi : f' \Rightarrow f$ in $\mathcal{E}$ is cartesian above a 2-cell $\beta : u' \Rightarrow u$ in $\mathcal{B}$ when the following 2-dimensional property holds:

for every 2-cell $\psi : f'' \Rightarrow f$

for every 2-cell $\beta' : u'' \Rightarrow u'$

such that $\beta' \circ \beta = p(\alpha)$

there exists

a unique 2-cell $\varphi' : f'' \Rightarrow f'$

such that $\varphi' \circ \varphi = \psi$

and $p(\varphi') = \beta'$.
Fibrations: the 2-categorical case

The second condition is satisfied by the 2-functor

\[ p : \mathcal{E} \rightarrow \mathcal{B} \]

when there exists a cartesian 2-cell

\[
\begin{array}{ccc}
\beta^* f & \xrightarrow{\varphi} & f \\
p & & p \\
\downarrow & & \downarrow \\
\beta & \xrightarrow{} & \beta
\end{array}
\]

for every morphism \( f \) in the fiber of \( u \) and for every 2-cell \( \beta : u' \Rightarrow u \) in \( \mathcal{B} \).
Indexed monads

Definition. An indexed monad is a pseudo 2-functor

\[ \mathcal{T} : \mathcal{B}^{\text{op}(1,2)} \longrightarrow \text{Mon} \]

from the basis 2-category \( \mathcal{B} \) to the 2-category \( \text{Mon} \) of monads.
A more intuitive view of indexed monads

Every indexed monad may be seen as an indexed category

\[ P : \mathcal{B}^{\text{op}(1,2)} \rightarrow \text{Mon} \rightarrow \text{Cat} \]

equipped with a monad

\[ T(b) : P(b) \rightarrow P(b) \]

on every fiber category \( P(b) \).
Starting point [LICS 2015]

The local state monad can be recovered from an indexed monad

$$\mathcal{T} : \mathcal{B}^{\text{op}(1,2)} \to \text{Mon}$$

where the 2-category $\mathcal{B}$ has natural numbers as objects, and

$$\mathcal{T}(n) : \text{Set} \to \text{Set}$$

is the global state monad on $n$ variables:

$$\mathcal{T}(n) = X \mapsto \text{Val}^n \Rightarrow \text{Val}^n \times X.$$

A 2-categorical account of Power's notion of block structure
First observation

**Theorem.** Every indexed monad

\[ \mathcal{T} : \mathbb{B}^{\text{op}(1,2)} \longrightarrow \text{Mon} \]

induces a 2-monad

\[ T : \mathcal{E} \longrightarrow \mathcal{E} \]

on the 2-category \( \mathcal{E} \) associated to the indexed category

\[ \mathcal{P} : \mathbb{B}^{\text{op}(1,2)} \longrightarrow \text{Mon} \longrightarrow \text{Cat} \]
Second observation

Moreover, the algebras of the indexed monad

\[ \mathcal{T} : \mathcal{B}^{\text{op}(1,2)} \to \text{Mon} \]

coincide with the algebras of the induced monad

\[ \text{Sect}(\mathcal{T}) : \text{Sect}(p) \to \text{Sect}(p) \]

on the category of sections of the fibration

\[ p : \mathcal{E} \to \mathcal{B} \]
Third observation

In the case of the local state monad

\[ L : [\text{Inj}, \text{Set}] \to [\text{Inj}, \text{Set}] \]

the category \([\text{Res}, \text{Set}]\) coincides with the category of sections of

\[ p : \mathcal{E} \to \mathcal{B} \]

associated to the indexed category

\[ \mathcal{P} : \mathcal{B}^{\text{op}(1,2)} \to \text{Mon} \to \text{Cat} \]

**Main benefit:** one recovers in this way the factorisation

\[ L = \ell^* \circ F \circ \exists \ell \]
Conclusion

This shows that one should probably replace presheaf categories $[\mathcal{C}^{\text{op}}, \text{Set}]$ by categories (or 2-categories) of sections of fibrations

$$p : E \rightarrow B$$

$\Rightarrow$ a theory of effect bundle which remains to be developed.
Short bibliography

PAM
Segal condition meets computational effects. LICS 2010.

PAM
Local states in string diagrams. RTA-TLCA 2014.

Kenji Maillard & PAM.
A fibrational account of local states. LICS 2015

Related works on local states


Soichiro Fujii, Shinya Katsumata & PAM.

Soichiro Fujii. Connections to his talk later in the session