A fibrational account of local states and beyond

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The local state monad

A monad with graded arities

Presheaf models

Key idea: interpret a type A as a family of sets

$$A_0 \qquad A_1 \qquad \cdots \qquad A_n \qquad \cdots$$

indexed by natural numbers, where each set

$$A_n$$

contains the programs of type A which have access to n variables.

Presheaf models

This defines a covariant presheaf

$$A_n \quad : \quad Inj \quad \longrightarrow \quad Set$$

on the category Inj of natural numbers and injections.

Every injection $f : p \longrightarrow q$ induces a function

$$A_f \quad : \quad A_p \quad \longrightarrow \quad A_q$$

obtained by renaming every register $i \in [p]$ by the register $f(i) \in [q]$.

Local stores [Plotkin & Power 2002]

The slightly intimidating monad

$$LA \quad : \quad n \quad \mapsto \quad S^n \quad \Rightarrow \quad \left(\int^{p \in Inj} \quad S^p \times A_p \times Inj(n,p) \quad \right)$$

on the presheaf category $[Inj, Set]$ where the contravariant presheaf

$$S^p \quad = \quad Val^p$$

describes the states available at degree p.

Graded arities

A graded arity is defined as a finite sum of representables

$$[\ p_0\,,\ \cdots\ ,\ p_n\,] \quad = \quad \underbrace{\langle 0\rangle + \cdots + \langle 0\rangle}_{p_0\ \text{times}} \ + \ \cdots \ + \ \underbrace{\langle n\rangle + \cdots + \langle n\rangle}_{p_n\ \text{times}}$$

where each representable presheaf

$$\langle n\rangle \quad : \quad Inj \quad \longrightarrow \quad Set$$

is defined as

$$\langle n\rangle \quad = \quad p \quad \mapsto \quad Inj(n,p)$$

Graded arities

This defines a full and faithful functor

$$\Sigma\, Inj^{op} \quad \longrightarrow \quad Inj^{op}$$

which generalizes the full and faithful functor

$$FinSet \quad \longrightarrow \quad Set$$

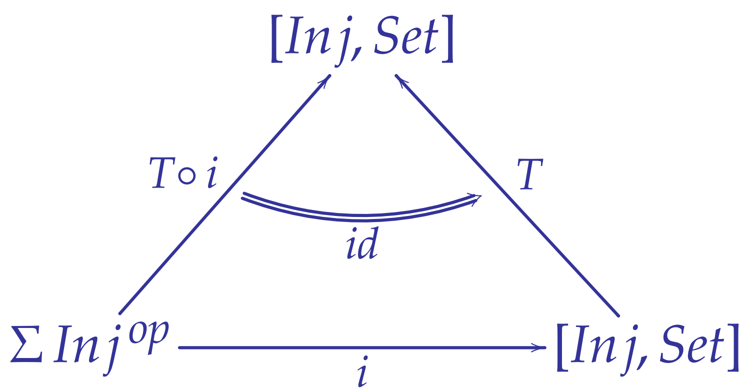
encountered in the case of finitary monads.

Monads with graded arities

Definition. A monad

$$T \quad : \quad [Inj, Set] \quad \longrightarrow \quad [Inj, Set]$$

has **graded arities** when the diagram



exhibits the functor T as a left Kan extension of $T \circ i$ along i.

The local state monad

Theorem [RTA-TLCA 2014] The local state monad

$$LA \quad : \quad n \quad \mapsto \quad S^n \quad \Rightarrow \quad \left(\int^{p \in Inj} \quad S^p \times A_p \times Inj(n,p) \quad \right)$$

is a monad with **graded arities**.

The local state monad

For that reason, the local state monad

$$L \quad : \quad [Inj, Set] \quad \longrightarrow \quad [Inj, Set]$$

is entirely described by its restriction

$$L \circ i \quad : \quad \Sigma\, Inj^{op} \quad \xrightarrow{i} \quad [Inj, Set] \quad \xrightarrow{L} \quad [Inj, Set]$$

to the subcategory $\Sigma\, Inj^{op}$ of graded arities.

Key observation behind this result

The local state monad

$$L \quad : \quad [Inj, Set] \quad \longrightarrow \quad [Inj, Set]$$

is the composite

$$L \quad = \quad \mathcal{F} \circ \mathcal{B}$$

of two monads with graded arities

$$\mathcal{F}, \mathcal{B} \quad : \quad [Inj, Set] \quad \longrightarrow \quad [Inj, Set]$$

related by a distributivity law

$$\lambda \quad : \quad \mathcal{B} \circ \mathcal{F} \quad \Rightarrow \quad \mathcal{F} \circ \mathcal{B}.$$

The fiber monad and the basis monad

The fiber monad

$$\mathcal{F} A \quad : \quad n \quad \mapsto \quad S^n \Rightarrow (S^n \times A_n)$$

is the **global state monad** applied on each fiber n while

$$\mathcal{B} = \ell^* \circ \exists_\ell \quad : \quad [Inj, Set] \quad \longrightarrow \quad [Inj, Set]$$

is the **change-of-basis monad** associated to a functor

$$\ell \quad : \quad Inj \quad \longrightarrow \quad Res.$$

An even nicer formulation

The functor $Inj \xrightarrow{\ell} Res$ induces an adjunction:

$$[Inj, Set] \underset{\ell^*}{\overset{\exists_\ell}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [Res, Set]$$

and the fiber monad lifts to a monad

$$\mathcal{F} \quad : \quad [Res, Set] \longrightarrow [Res, Set]$$

Factorisation theorem [RTA-TLCA 2014]

The local state monad L coincides with the monad $\ell^* \circ \mathcal{F} \circ \exists_\ell$.

What does this factorisation tell us?

Lookup ← Update ← Discharge ← Allocate ← Permute

Local mnemoids

Definition. A **local mnemoid** is a family of sets

$$A_0 \qquad A_1 \qquad \cdots \qquad A_n \qquad \cdots$$

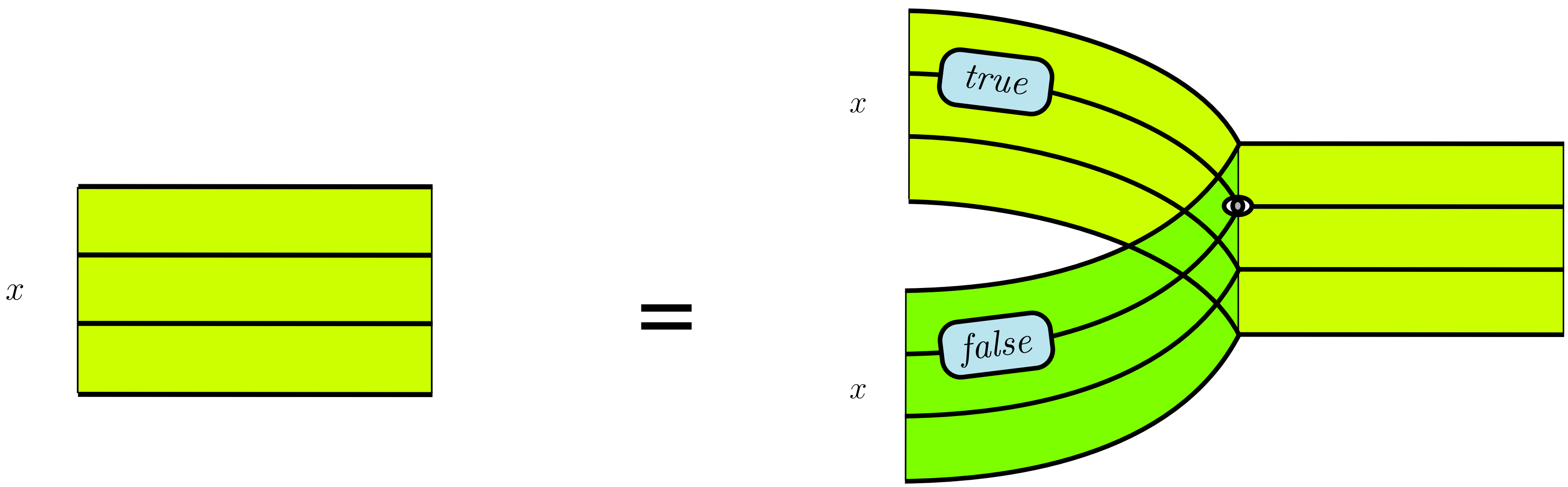
equipped with the following operations

$$\begin{array}{rcccc}
\texttt{lookup}_{\langle loc \rangle} & : & A_n \times A_n & \longrightarrow & A_n \\
\texttt{update}_{\langle loc,val \rangle} & : & A_n & \longrightarrow & A_n \\
\texttt{fresh}_{\langle loc,val \rangle} & : & A_{n+1} & \longrightarrow & A_n \\
\texttt{dispose}_{\langle loc \rangle} & : & A_n & \longrightarrow & A_{n+1} \\
\texttt{permute}_{\langle loc \rangle} & : & A_n & \longrightarrow & A_n
\end{array}$$

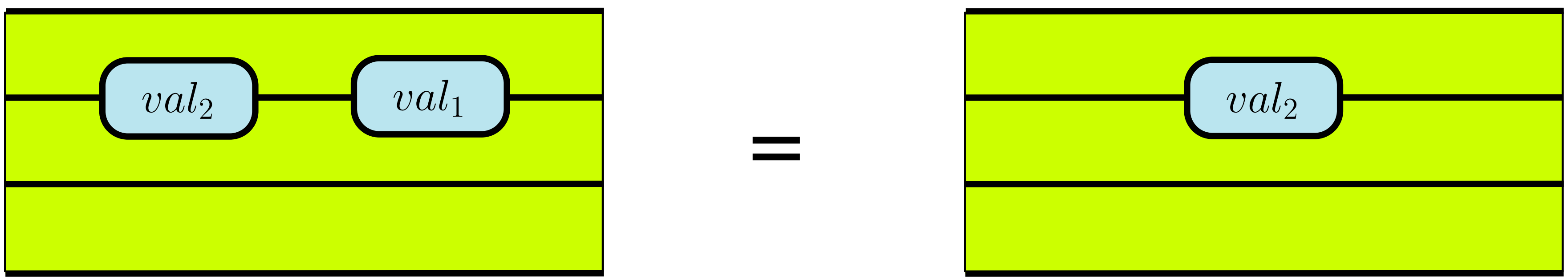
satisfying a series of elementary equations.

An algebraic presentation of the fiber monad

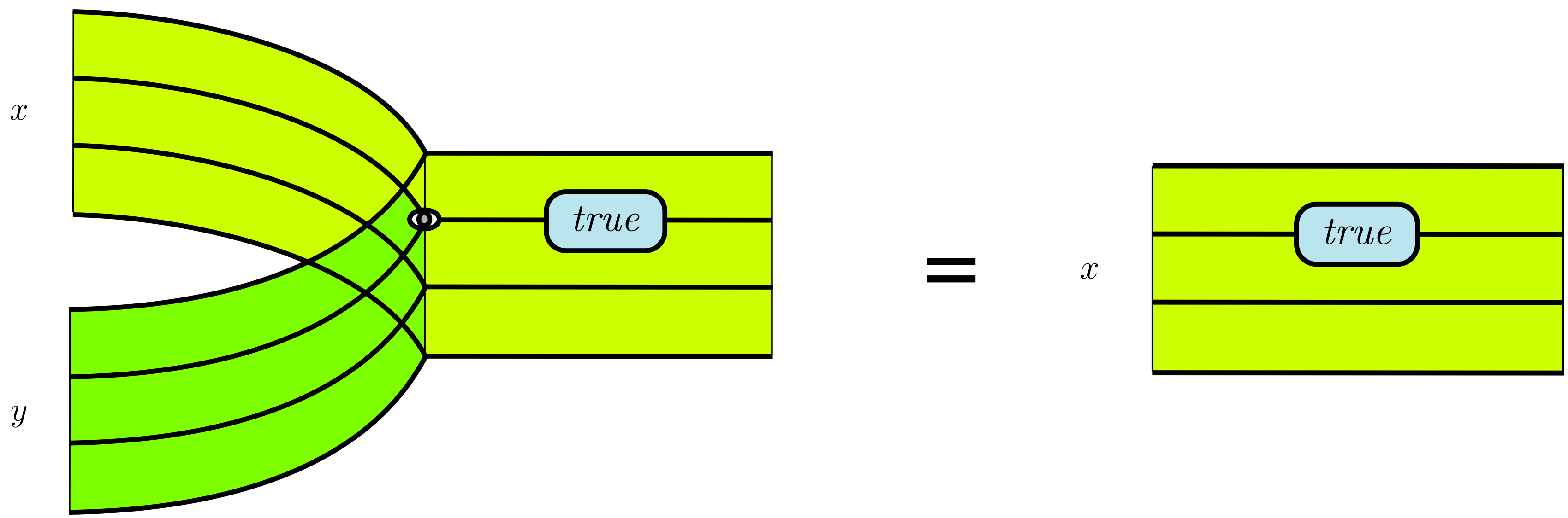
Creation lookup – update



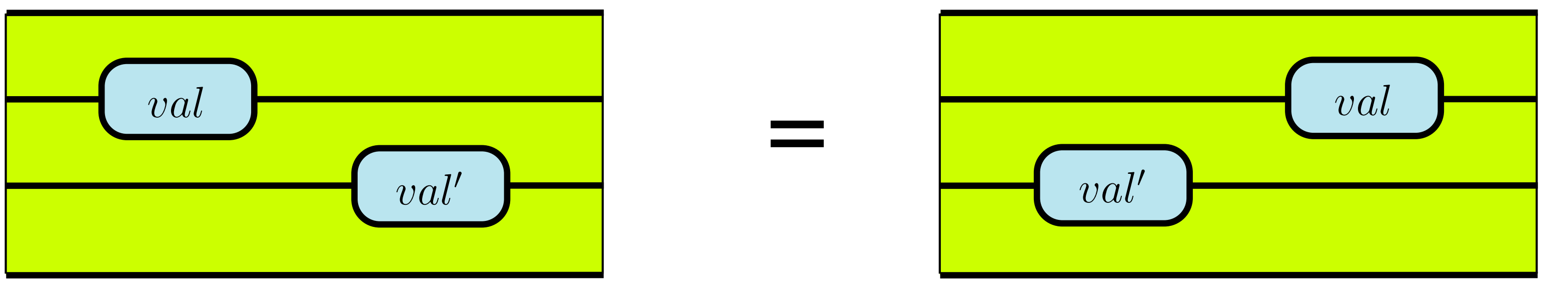
Interaction update – update



Interaction update – lookup

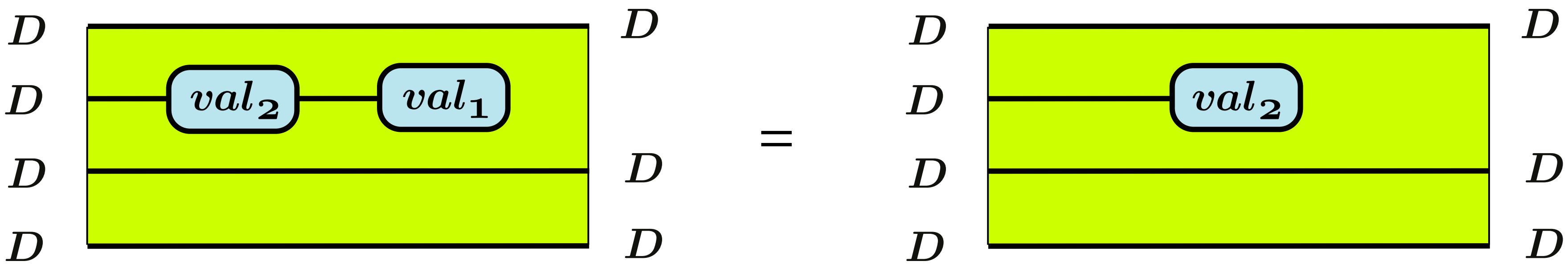


Commutation update – update

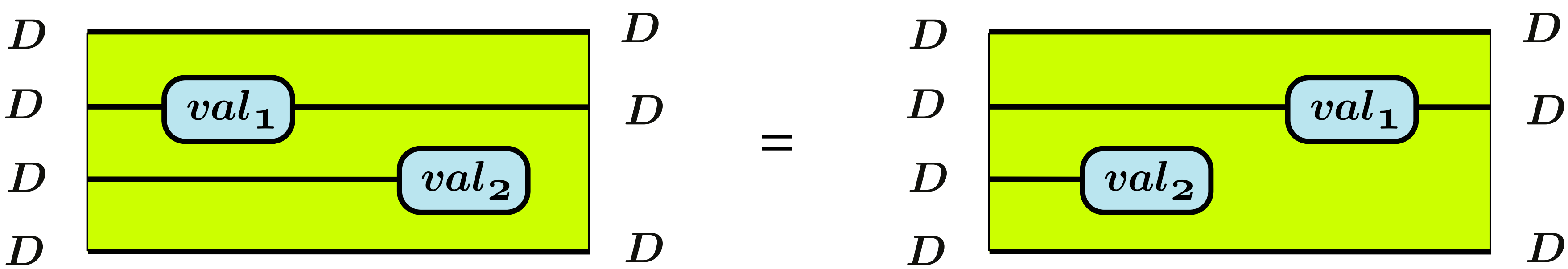


An algebraic presentation of the basis monad

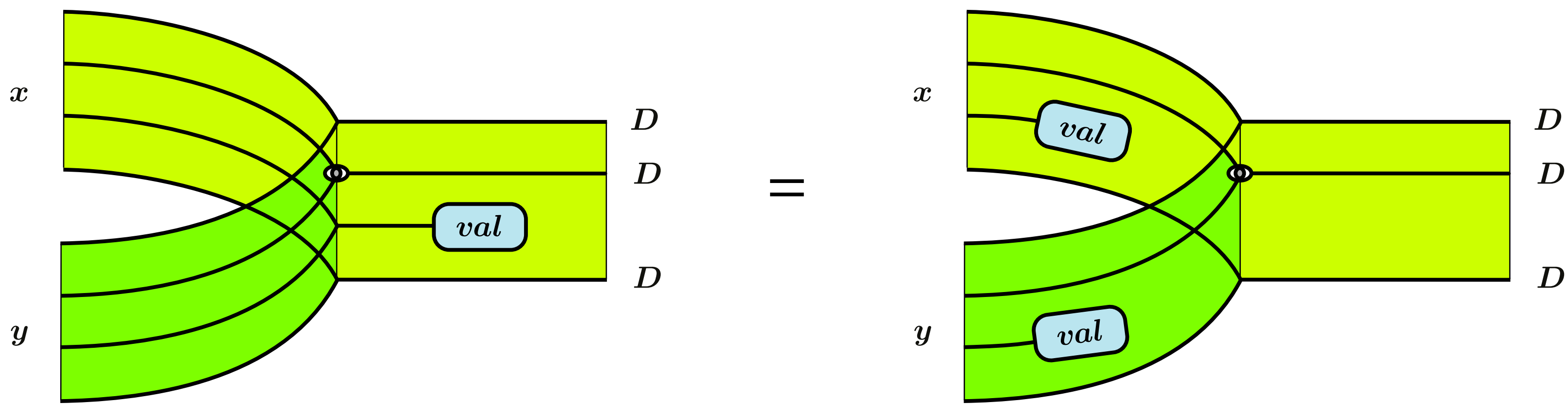
Interaction fresh – update



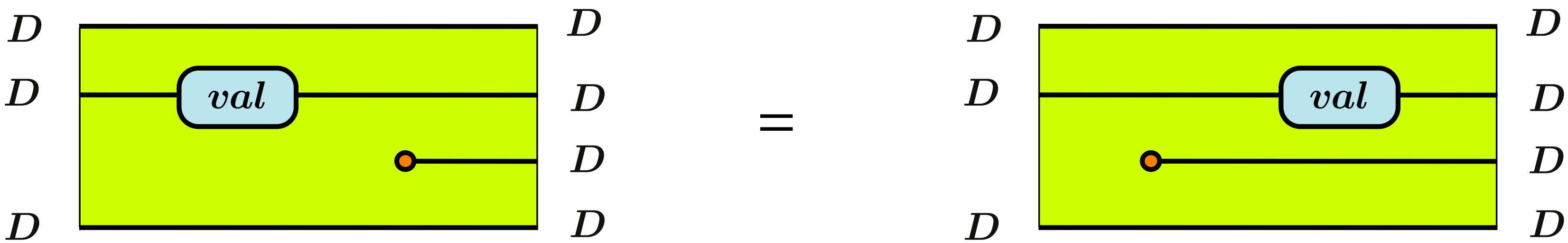
Commutation fresh – update



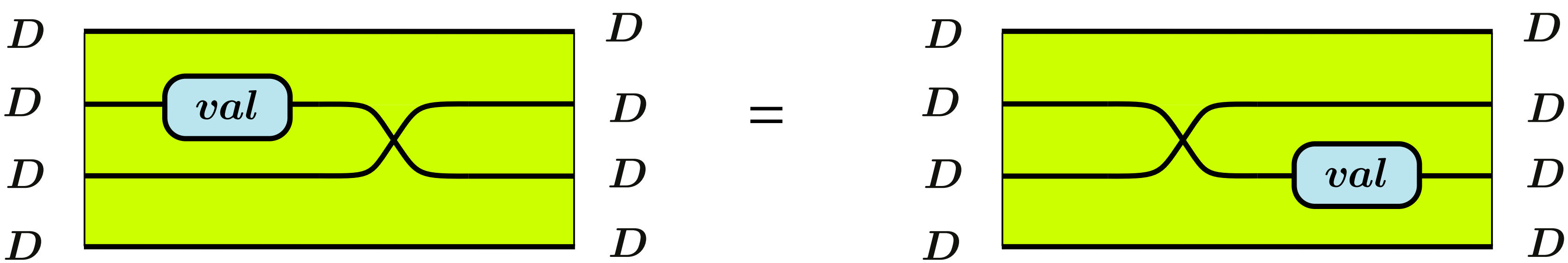
Commutation fresh – update



Interaction collect – update



Interaction permute – lookup



Local mnemoids

Definition. A **local mnemoid** is a family of sets

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\mathtt{dispose}_{\langle loc \rangle} & : & A_n & \longrightarrow & A_{n+1} \\
\mathtt{permute}_{\langle loc \rangle} & : & A_n & \longrightarrow & A_n
\end{array}$$

satisfying the equations above.

An algebraic presentation of the local state monad [RTA-TLCA 2014]

the category of local mnemoids

is equivalent to

the category of algebras of the local state monad

Lookup $\leftarrow$ Update $\leftarrow$ Discharge $\leftarrow$ Allocate $\leftarrow$ Permute

The Grothendieck construction

A fibrational way to glue effects

The Grothendieck construction

An indexed $\mathscr{B}$-category is a pseudo functor from a category $\mathscr{B}$

$$\mathscr{P} \quad : \quad \mathscr{B}^{op} \longrightarrow \mathbf{Cat}$$

which associates to every $b \in \mathscr{B}$ a category $\mathscr{P}(b)$.

The Grothendieck construction "glues" together the $\mathscr{P}(b)$'s into a fibration

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

whose fiber

$$\mathscr{E}_b \quad = \quad p^{-1}(b)$$

above the object $b \in \mathscr{B}$ coincides with the category $\mathscr{P}(b)$.

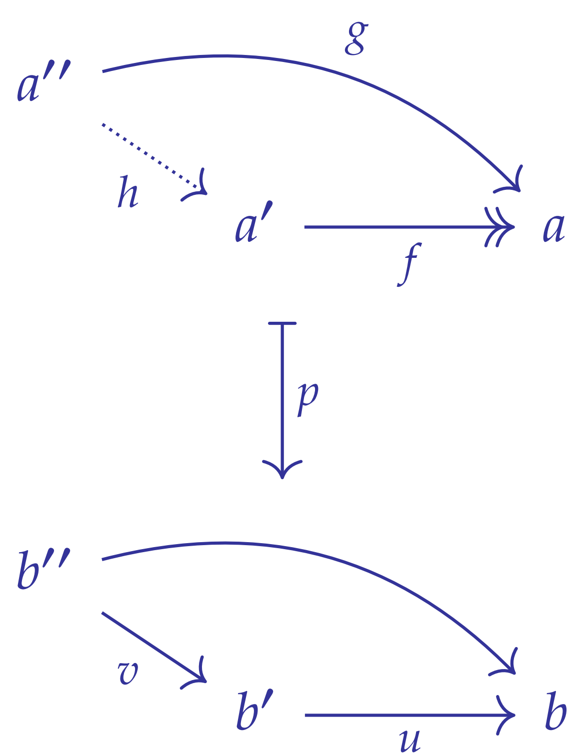
Cartesian morphisms

A morphism $f : a' \to a$ in $\mathscr{E}$ is cartesian above $u : b' \to b$ in $\mathscr{B}$ when the following property holds:

for every morphism $g : a'' \to a$

for every morphism $v : b'' \to b'$
such that $u \circ v = p(g)$

there exists
a unique morphism $h : a'' \to a'$
such that $f \circ h = g$
and $p(h) = v$.



Fibrations

Definition. A fibration is a functor

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

when there exists a cartesian morphism

$$\begin{array}{ccc} u^*a & \xrightarrow{\quad f \quad} & a \\ & \big\downarrow{\scriptstyle p} & \\ b' & \xrightarrow[\quad u \quad]{} & b \end{array}$$

for every object a in the fiber of b and every morphism $u : b' \to b$.

The Grothendieck construction

The construction can be adapted to a pseudo 2-functor

$$\mathscr{P} \quad : \quad \mathscr{B}^{op(1,2)} \longrightarrow \mathbf{Cat}$$

from a 2-category $\mathscr{B}$.

One obtains in that case a 2-functor

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

satisfying the following fibrational properties.

Fibrations: the 2-categorical case

Extended definition. A fibration is a 2-functor

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

satisfying the following properties:

1. the underlying functor is a fibration (in the usual sense)

2. the functor

$$p(a,a') \quad : \quad \mathscr{E}(a,a') \longrightarrow \mathscr{B}(p(a),p(a'))$$

is a discrete fibration for every $a,a' \in \mathscr{E}$.

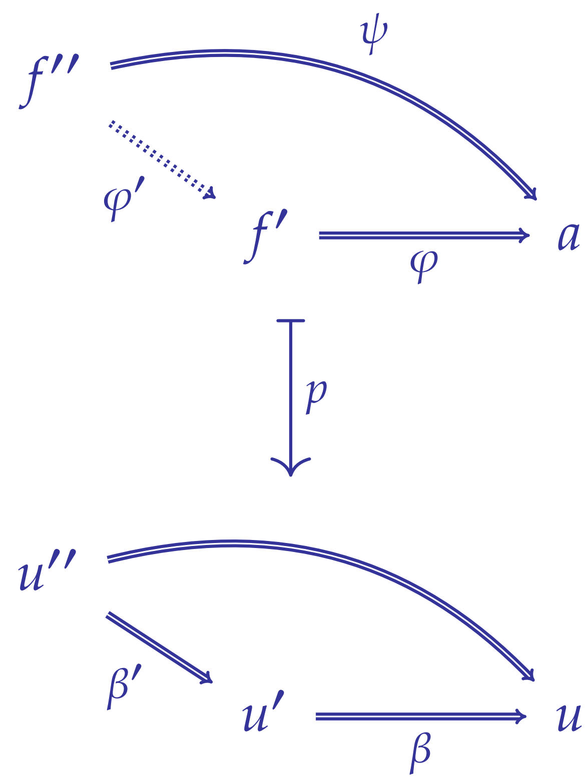
Cartesian 2-cells

A 2-cell $\varphi : f' \Rightarrow f$ in $\mathscr{E}$ is cartesian above a 2-cell $\beta : u' \Rightarrow u$ in $\mathscr{B}$ when the following 2-dimensional property holds:

for every 2-cell $\psi : f'' \to f$

for every 2-cell $\beta' : u'' \to u'$
such that $\beta' \circ \beta = p(\alpha)$

there exists
a unique 2-cell $\varphi' : f'' \to f'$
such that $\varphi' \circ \varphi = \psi$
and $p(\varphi') = \beta'$.



Fibrations: the 2-categorical case

The second condition is satisfied by the 2-functor

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

when there exists a cartesian 2-cell

$$\begin{array}{ccc} \beta^* f & \overset{\varphi}{\Longrightarrow} & f \\ & \Big\downarrow p & \\ u' & \underset{\beta}{\Longrightarrow} & u \end{array}$$

for every morphism f in the fiber of u and for every 2-cell $\beta : u' \Rightarrow u$ in $\mathscr{B}$.

Indexed monads

Definition. An indexed monad is a pseudo 2-functor

$$\mathscr{T} \quad : \quad \mathscr{B}^{op(1,2)} \longrightarrow \mathbf{Mon}$$

from the basis 2-category $\mathscr{B}$ to the 2-category $\mathbf{Mon}$ of monads.

A more intuitive view of indexed monads

Every indexed monad may be seen as an indexed category

$$\mathscr{P} \quad : \quad \mathscr{B}^{op(1,2)} \longrightarrow \mathbf{Mon} \longrightarrow \mathbf{Cat}$$

equipped with a monad

$$\mathscr{T}(b) \quad : \quad \mathscr{P}(b) \longrightarrow \mathscr{P}(b)$$

on every fiber category $\mathscr{P}(b)$.

Starting point [LICS 2015]

The local state monad can be recovered from an indexed monad

$$\mathscr{T} \quad : \quad \mathscr{B}^{op(1,2)} \longrightarrow \mathbf{Mon}$$

where the 2-category $\mathscr{B}$ has natural numbers as objects, and

$$\mathscr{T}(n) \quad : \quad Set \longrightarrow Set$$

is the global state monad on n variables:

$$\mathscr{T}(n) \quad = \quad X \quad \mapsto \quad Val^n \Rightarrow Val^n \times X.$$

A 2-categorical account of Power's notion of block structure

First observation

Theorem. Every indexed monad

$$\mathscr{T} \quad : \quad \mathscr{B}^{op(1,2)} \longrightarrow \mathbf{Mon}$$

induces a 2-monad

$$T \quad : \quad \mathscr{E} \longrightarrow \mathscr{E}$$

on the 2-category $\mathscr{E}$ associated to the indexed category

$$\mathscr{P} \quad : \quad \mathscr{B}^{op(1,2)} \longrightarrow \mathbf{Mon} \longrightarrow \mathbf{Cat}$$

Second observation

Moreover, the algebras of the indexed monad

$$\mathscr{T} \quad : \quad \mathscr{B}^{op(1,2)} \longrightarrow \mathbf{Mon}$$

coincide with the algebras of the induced monad

$$Sect(T) \quad : \quad Sect(p) \longrightarrow Sect(p)$$

on the category of **sections** of the fibration

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

Third observation

In the case of the local state monad

$$L \quad : \quad [Inj, Set] \longrightarrow [Inj, Set]$$

the category $[Res, Set]$ coincides with the category of sections of

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

associated to the indexed category

$$\mathscr{P} \quad : \quad \mathscr{B}^{op(1,2)} \longrightarrow \mathbf{Mon} \longrightarrow \mathbf{Cat}$$

Main benefit: one recovers in this way the factorisation

$$L \quad = \quad \ell^* \circ \mathcal{F} \circ \exists_\ell$$

Conclusion

This shows that one should probably replace presheaf categories

$$[\mathscr{C}^{op}, Set]$$

by categories (or 2-categories) of sections of fibrations

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

$\Rightarrow$ a theory of effect bundle which remains to be developed.

Short bibliography

PAM
Segal condition meets computational effects. LICS 2010.

PAM
Local states in string diagrams. RTA-TLCA 2014.

Kenji Maillard & PAM.
A fibrational account of local states. LICS 2015

Related works on local states

John Power. Indexed Lawvere theories for local state. CRM 2011.

Soichiro Fujii, Shinya Katsumata & PAM.
Towards a formal theory of graded monads. Fossacs 2016.

Soichiro Fujii. Connections to his talk later in the session