Flexible satisfaction

LCC 2016, Aix-Marseille Université

Marcel Jackson
LA TROBE UNIVERSITY

A problem in semigroup theory

The semigroup $\mathbf{B}_2^1$

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

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Theorem (J, 2015)

It is $\mathtt{NP}$-hard to decide membership of a finite semigroup in $\mathsf{HSP}(\mathbf{B}_2^1)$

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Theorem (J, 2015)

It is $\mathtt{NP}$-hard to decide membership of a finite semigroup in $\mathrm{HSP}(\mathbf{B}_2^1)$

Requires the $\mathtt{NP}$-hardness of a promise problem $+$1-in-3SAT:

NO: *a NO instance of +1-in-3SAT*
YES: *the* $+$*1-in-3SAT instance lies in the "quasivariety" of* $+$*1-in-3SAT*

Overview of talk and results

- Quasivariety concepts, and their relationship with constraint problems and implicit constraints

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  - For a fixed template CSP: absence of implicit constraints coincides with membership in quasivariety
  - implementation of implicit constraints coincides with “reflection” into the quasivariety
- Result: Intractability of the stated promise problem $+1$-in-3SAT
- Result: Generalisation to (probably) all finite core relational structures with hard CSP

Quasivarieties (Maltsev)

A. I. Maltsev

Quasivarieties (Maltsev)

Semantic

Closed under isomorphic copies of substructures, direct products and ultraproducts

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Syntactic

Definable by a set of quasi-equations:

$$\Big(\mathop{\&}_{1 \leq i \leq n} \alpha_i \Big) \to \alpha_0$$

with each $\alpha_0, \ldots, \alpha_n$ atomic

Universal Horn classes

Semantic

Closed under isomorphic copies of substructures, direct products over nonempty index sets and ultraproducts

Syntactic

Definable by a set of universal Horn sentences:

$$\bigvee_{0 \leq i \leq n} \alpha_i$$

where all α_i are atomic or negated atomic, and at most one is atomic

Quasivarieties versus universal Horn classes

- The quasivariety $\mathrm{Q}(K)$ generated by a universal Horn class K differs from K, if at all, only on one element structures

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$\Rightarrow$ We can blur the distinction when it suits us

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Cancelative semigroups:

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Cancelative semigroups: $x(yz) = (xy)z$,
$xy = xz \to y = z, \qquad xy = zy \to x = z$

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Semigroups embeddable in a group:

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Semigroups embeddable in a group: *it's complicated*

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Examples

Ordered sets:

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Examples

Ordered sets: $x \leq x$, $\quad x \leq y \ \& \ y \leq z \to x \leq z$, $x \leq y \ \& \ y \leq x \to x = y$

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Simple graphs:

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Examples

Simple graphs: $x \not\sim x$, $x \sim y \to y \sim x$ (a universal Horn class)

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2-colourable graphs:

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2-colourable graphs: simple graphs with no odd cycles (a universal Horn class)

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3-colourable graphs:

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3-colourable graphs: . . . (a universal Horn class)

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Definable by a set of quasi-equations:

$$\Big(\mathop{\&}_{1 \le i \le n} \alpha_i \Big) \rightarrow \alpha_0$$

Examples

For any fixed finite relational structure $\mathbb{A}$: the class of structures admitting a homomorphism into $\mathbb{A}$ (a universal Horn class)

Constraint Satisfaction Problems

Fix a relational structure $\mathbb{A}$ (always finite, with signature $\mathcal{R}$)

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(Alternatively, the elements of the universe B are the "variables" and the tuples $(b_1, \ldots, b_i) \in r$ for $r \in \mathcal{R}^{\mathbb{B}}$ are the "constraints")

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Example: +1-in-3SAT

$$\mathbb{A} = \langle \{0,1\}; \overbrace{\{(0,0,1),(0,1,0),(1,0,0)\}}^{\mathcal{R}} \rangle$$

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Very important

CSP($\mathbb{A}$) is a quasivariety membership problem, but it is not membership in the quasivariety of $\mathbb{A}$

CSP($\mathbb{A}$) versus $Q(\mathbb{A})$

Example: signature $\{\sim, 0, 1\}$

$\mathbb{R}$

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- CSP($\mathbb{R}$) is directed graph unreachability: $\mathtt{NL}$-complete

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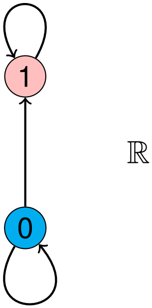
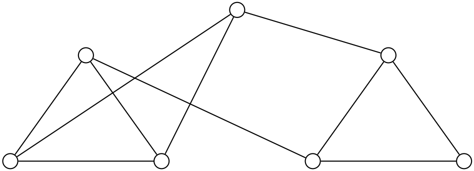


- CSP($\mathbb{R}$) is directed graph unreachability: $\mathtt{NL}$-complete
- $Q(\mathbb{R})$ is (roughly) just the class of ordered sets (first order definable)

CSP($\mathbb{A}$) versus $Q(\mathbb{A})$: core retracts



A graph

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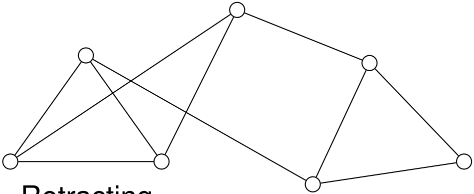

Retracting…

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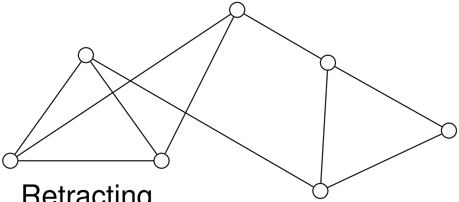

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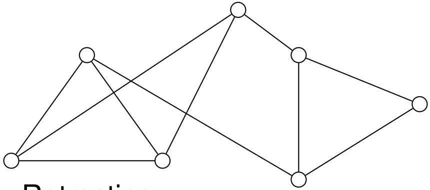

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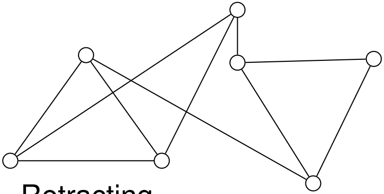

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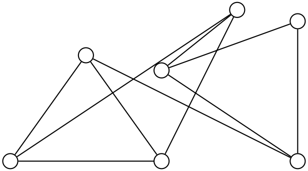

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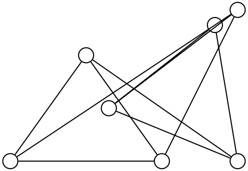

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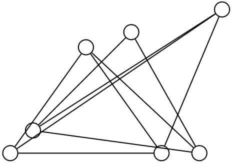

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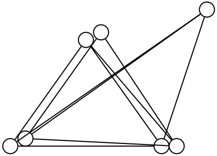

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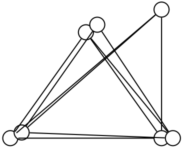

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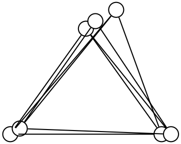

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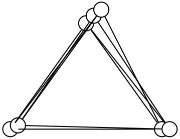

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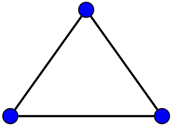
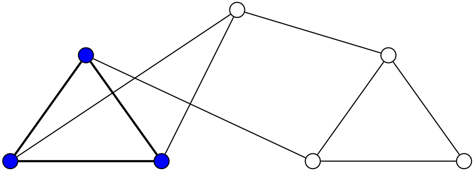


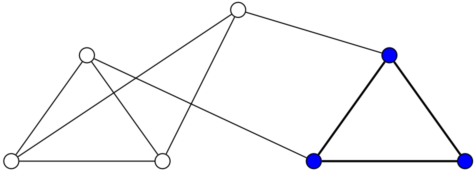
A core retract: all endomorphisms are automorphisms

CSP($\mathbb{A}$) versus $Q(\mathbb{A})$: core retracts



The original graph, with its core retract in bold

CSP($\mathbb{A}$) versus $Q(\mathbb{A})$: core retracts



Another core retract, necessarily isomorphic

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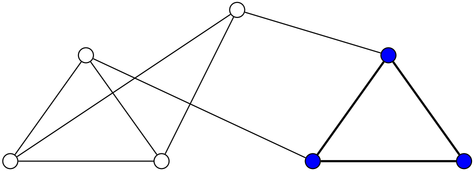

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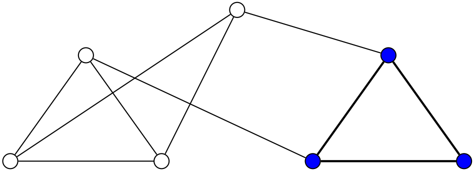
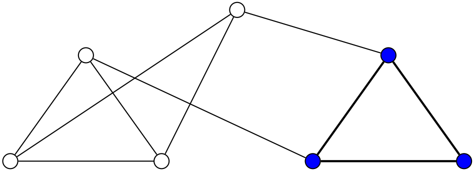


Another core retract, necessarily isomorphic

- *It is obvious that* $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathrm{core}(\mathbb{A}))$
- Homomorphism equivalent structures have the same CSP, *but not necessarily the same quasivariety*

CSP($\mathbb{A}$) versus $Q(\mathbb{A})$: core retracts



Another core retract, necessarily isomorphic

- *It is obvious that* $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathrm{core}(\mathbb{A}))$
- In particular: $Q(\mathbb{A})$ is almost never equal to $Q(\mathrm{core}(\mathbb{A}))$

Implicit constraints for CSP($\mathbb{A}$)

Frozen in

For a finite $\mathcal{R}$-structure $\mathbb{B}$, a relation $r \in \mathcal{R} \cup \{=\}$, a tuple $(b_1, \ldots, b_n)$ is *frozen in* to r if every homomorphism $\phi : \mathbb{B} \to \mathbb{A}$ has $(\phi(b_1), \ldots, \phi(b_n)) \in r$

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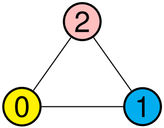
Example: $\mathcal{R} = \{\sim\}$ (binary)



$\mathbb{K}_3$

(here $\mathbb{A}$ will be $\mathbb{K}_3$)

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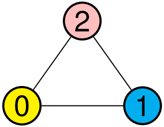
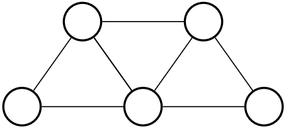


$\mathbb{K}_3$



$\mathbb{B}$

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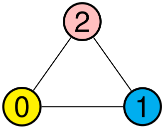

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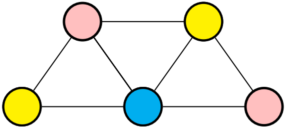

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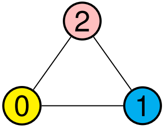

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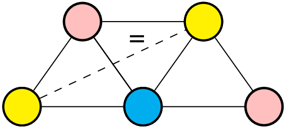

$\mathbb{B}$

- *Frozen-in* to $=$ (may as well have been equal)

Implicit constraints for CSP($\mathbb{A}$)

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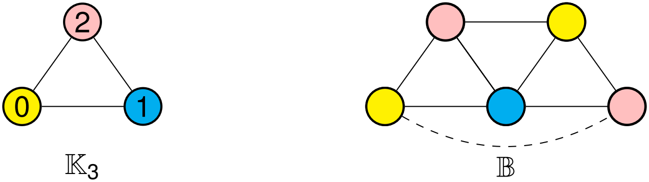
Example: $\mathcal{R} = \{\sim\}$ (binary)



- *Frozen-in* to the edge relation (may as well have added the edge)

Implicit constraints for CSP($\mathbb{A}$)

Frozen in

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The backbone of $\mathbb{B}$ (wrt $\mathbb{A}$) is the subset of B that are forced to take only one value under any homomorphism from $\mathbb{B}$ to $\mathbb{A}$

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Emergence of large backbone a possible cause of intractability: "efficient means of finding the backbone would be specific to each problem type, but should nonetheless provide a step forwards in algorithm efficiency." (Monasson et al, Nature, 1999)

Implicit constraints for CSP($\mathbb{A}$)

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(Previous talk (Ham): backbone cannot be cause of intractability)

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(also, Beacham 2000 and others)

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r is unfrozen:

if no tuple is frozen in to r

Unfrozenness is quasivariety membership!

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Theorem (J, 2015, after Maltsev)

For a finite relational structure $\mathbb{A}$ and a subset $\mathcal{S} \subseteq \mathcal{R} \cup \{=\}$. TFAE for a finite relational structure $\mathbb{B}$ having at least one homomorphism into $\mathbb{A}$:

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In particular:

no implicit constraints $\Leftrightarrow$ in the quasivariety of $\mathbb{A}$

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In particular:

no relations required to be unfrozen $\Leftrightarrow$ in CSP($\mathbb{A}$)

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In particular:

equality is unfrozen $\Leftrightarrow$ in SEP($\mathbb{A}$)

+1-in-3SAT

Recall:

Example: +1-in-3SAT

$\mathbb{A} = \langle \{0, 1\}; \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\} \rangle$

+1-in-3SAT

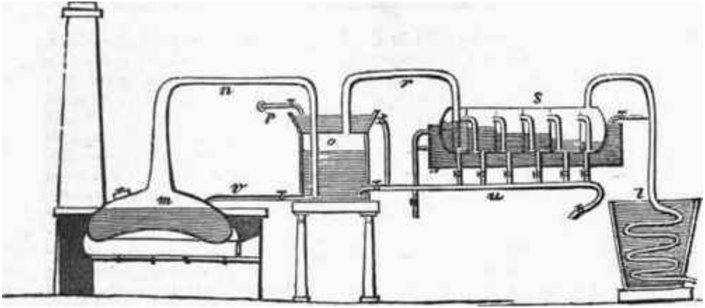
Recall:

Example: +1-in-3SAT

$\mathbb{A} = \langle \{0,1\}; \{(0,0,1),(0,1,0),(1,0,0)\} \rangle$

- $\rightarrow$ original semigroup theoretic motivation
- $\rightarrow$ fulcrum for Ham's Gap Trichotomy Theorem on Boolean domains

+1-in-3SAT



Proof by “distillation”
(passing the problem through a series of otherwise unnecessary reductions)

+1-in-3SAT

NAE3SAT $\leq_{\mathrm{L}}$ NAE21SAT

Introduces "6-robustness" promise:
either there is no solution or every assignment on at most 6 variables extends to a solution

+1-in-3SAT

NAE3SAT $\leq_{\mathrm{L}}$ NAE21SAT

Introduces “6-robustness” promise:
either there is no solution or every assignment on at most 6 *variables extends to a solution*

Follows technique of Gottlob, CP 2011

+1-in-3SAT

NAE3SAT $\leq_L$ NAE21SAT

$\leq_L$ NAE3SAT

(Standard reduction.) Maintains “2-robustness” promise:
either there is no solution or every assignment on at most 2 variables extends to a solution

+1-in-3SAT

$$\begin{aligned}\text{NAE3SAT} &\leq_{\mathrm{L}} \text{NAE21SAT}\\ &\leq_{\mathrm{L}} \text{NAE3SAT}\\ &\leq_{\mathrm{L}} \text{G3C on graphs with triangulated edges}\end{aligned}$$

(Standard reduction, but with triangulation of all edges.) Maintains "2-robustness" promise:
either there is no 3-colouring, or every valid colouring of any two vertices extends to a 3-colouring

+1-in-3SAT

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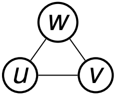
(Non-triangulated case proved by Abramsky, Gottlob and Kolaitis, IJCAI 2013)

+1-in-3SAT

$$\begin{aligned}\text{NAE3SAT} &\leq_{\mathrm{L}} \text{NAE21SAT}\\ &\leq_{\mathrm{L}} \text{NAE3SAT}\\ &\leq_{\mathrm{L}} \text{G3C on graphs with triangulated edges}\\ &\leq_{\mathrm{L}} \text{+1-in-3SAT}\end{aligned}$$

Obvious reduction: for each triangle 

and each "colour" $i \in \{0, 1, 2\}$ create clauses

$(u_i \vee v_i \vee w_i)$: *"colour i appears exactly once"* and

$(u_0 \vee u_1 \vee u_2)$: *"vertex u has exactly one colour"*

Mega extension

Big Gap Theorem (previous talk), enables us to prove a corresponding result for probably any hard CSP

Mega extension

Quasivariety Gap Theorem (Ham and Jackson, 2016)

For a finite relational structure $\mathbb{A}$ of finite type:

-

Mega extension

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For a finite relational structure $\mathbb{A}$ of finite type:

- if the core retract of $\mathbb{A}$ has no “weak NU”, then deciding membership in the quasivariety of $\mathbb{A}$ is $\mathtt{NP}$-complete

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- “weak NU” *is a polymorphism property*
- *Algebraic Dichotomy Conjecture: a CSP over core $\mathbb{A}$ is $\mathtt{NP}$-complete if and only if it has no weak NU (Bulatov, Jeavons, Krokhin, 2001)*

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  NO $\mathbb{B}$ has no homomorphisms into $\mathbb{A}$ (or core($\mathbb{A}$))

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A slightly nontrivial consequence of the Big Gap Theorem (previous talk)

Corollary

For a finite simple graph $\mathbb{G}$:

1. $\mathbb{G}$ bipartite: membership in $Q(\mathbb{G})$ is tractable
2. $\mathbb{G}$ not bipartite: membership in $Q(\mathbb{G})$ is $\mathtt{NP}$-complete

Corollary

For a finite simple graph $\mathbb{G}$:

1. $\mathbb{G}$ bipartite: deciding $CSP(\mathbb{G})$ is tractable
2. $\mathbb{G}$ not bipartite: deciding $CSP(\mathbb{G})$ is $\mathtt{NP}$-complete

(The Hell-Nešetřil dichotomy)

Corollary

For a finite simple graph $\mathbb{G}$:

1. $\mathbb{G}$ bipartite: Both $Q(\mathbb{G})$ and $\mathrm{CSP}(G)$ are tractable
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Corollary

Let $\mathbb{A}$ be a relational structure on $\{0, 1\}$. Deciding membership in the quasivariety of $\mathbb{A}$ is

- solvable in polynomial time if $\mathbb{A}$ has a weak NU
- $\mathtt{NP}$-complete otherwise

Fundamental ideas for proof

1. all of the previous talk (Big Gap Theorem)

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reflection?

enables transition from $(k, \mathcal{F})$-robust satisfiability to quasivariety membership

Reflection into a quasivariety (Maltsev again)

Semantic: given generator $\mathbb{A}$ for quasivariety

Apply natural homomorphism from $\mathbb{B}$ into the direct product $\prod_{\text{hom}(\mathbb{B},\mathbb{A})} \mathbb{A}$ (ie $b(\phi) = \phi(b)$)

- The reflection of $\mathbb{B}$ into $\mathsf{Q}(K)$ is the induced substructure on the image of $\mathbb{B}$

Blah blah blah blah[1]

[1] actually it's really beautiful

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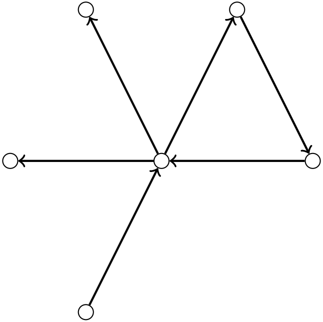
Syntactic: given set Σ of axioms for quasivariety

Iteratively apply quasi-equations in Σ to $\mathbb{B}$: for each interpretation of the premise, implement the conclusion

Example: syntactic reflection into ordered sets



A directed graph. . .

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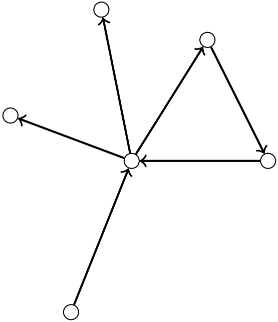

(Redrawing only. . .)

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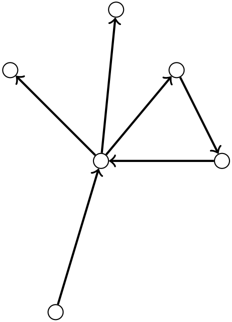

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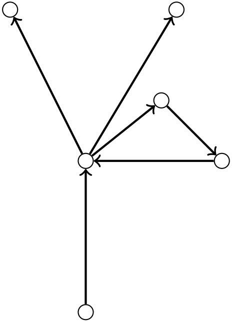

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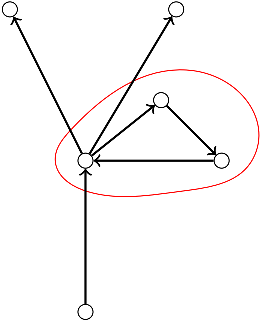

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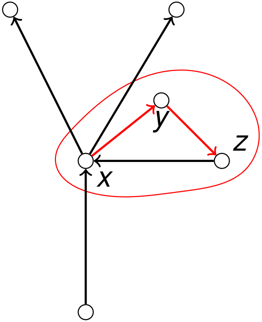
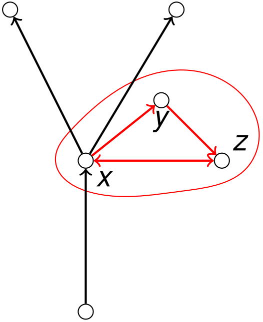


Reflecting

$x \leq y \ \& \ y \leq z \rightarrow x \leq z$

Example: syntactic reflection into ordered sets



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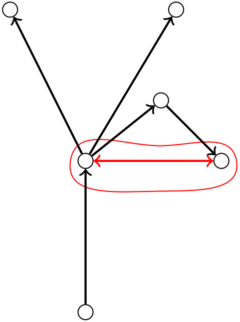

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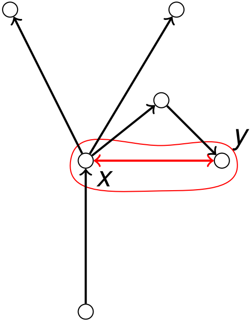

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$$x \leq y \;\; \& \;\; y \leq x \to x = y$$

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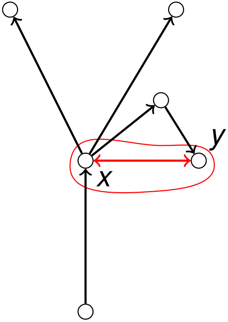

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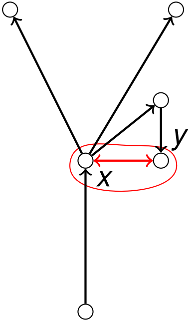

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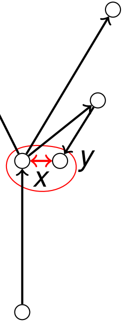

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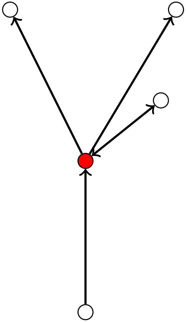
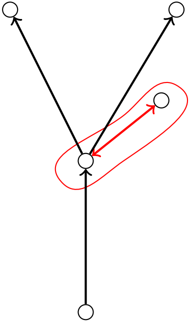
Example: syntactic reflection into ordered sets



Reflecting (ignoring loops)

$$x \leq y \ \& \ y \leq x \rightarrow x = y$$

Example: syntactic reflection into ordered sets



Reflecting

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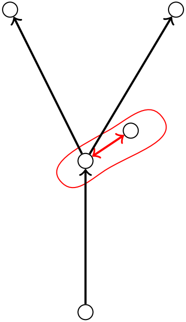
Example: syntactic reflection into ordered sets



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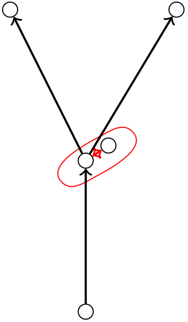

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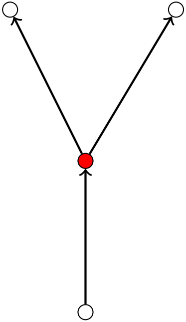

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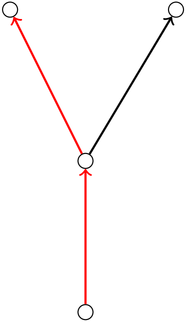

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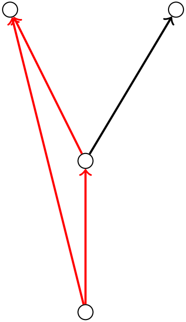

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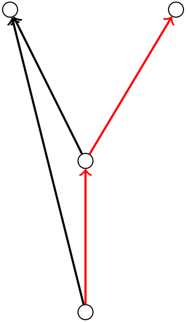

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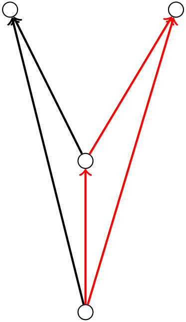

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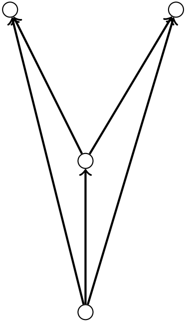

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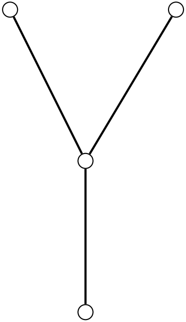

As a Hasse Diagram

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(he can't get no satisfaction)

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Apply this first order reduction from the promise problem in the Big Gap Theorem to obtain the Quasivariety Gap Theorem

Thank you!