Distributive Traced Categories

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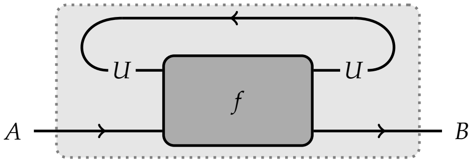
Traced Categories and the Int construction

(a reminder)

Traced categories

Trace in a symmetric monoidal category: operation turning $f : A \oplus U \to B \oplus U$ into $\mathbf{Tr}^U[f] : A \to B$.



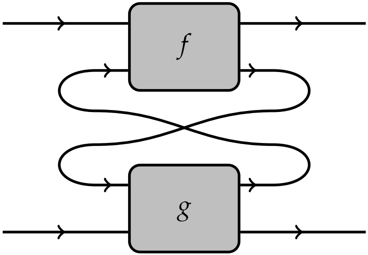
Understood as a *feedback along* U. Various natural properties.

Key concept of the categorical approach to Geometry of Interaction.

Comes with notions of *traced functor*, *monoidal natural transformation* preserving relevant structure.

Int construction

Given a traced category $\mathfrak{C}$, define $\mathbf{Int}(\mathfrak{C})$ based on the idea of *interactive evaluation*.



$\mathbf{Int}(\mathfrak{C})$ is a *monoidal closed category*, can interpret (linear) λ-calculus.

Promotion functors

Extra structure is needed to handle structural rules (duplication, erasure).

This is achieved via the notion of *promotion functor*: a functor $\mathtt{!}$ that is traced, with retractions (embedding-inverse)

- *digging* retraction $\mathtt{t}_A, \mathtt{t}'_A : \mathtt{!!}A \lhd \mathtt{!}A$
- *dereliction* retraction $\mathtt{d}_A, \mathtt{d}'_A : A \lhd \mathtt{!}A$
- *contraction* retraction $\mathtt{c}_A, \mathtt{c}'_A : \mathtt{!}A \oplus \mathtt{!}A \lhd \mathtt{!}A$
- *weakening* retraction $\mathtt{w}_A, \mathtt{w}'_A : 0 \lhd \mathtt{!}A$

In a traced category with a promotion functor, one can interpret full linear logic, λ-calculus, *etc.*

Examples of promotion functors

In the category of *sets and partial injections*: $!A = \mathbb{N} \times A$, with the retractions defined using encodings of $\mathbb{N} + \mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$.

In Girard's operator model: $!A = \mathcal{H} \otimes A$, ($\mathcal{H}$ the standard Hilbert space) with the retractions defined using embeddings of $\mathcal{H} \oplus \mathcal{H}$ and $\mathcal{H} \otimes \mathcal{H}$ into $\mathcal{H}$.

In Schöpp-Dal Lago's `IntML`: $!_X A = X \times A$ with available retractions depending on X.

There seems to be a general pattern, let's try to axiomatize it.

Distributive Traced Categories

Definitions

In all examples, the same structure shows up: we have a monoidal product $(+, \oplus, \dots)$ and another product $(\times, \otimes, \dots)$ that distributes over it.

There is a categorical notion for that, a *rig category*: two monoidal products $\oplus, \bullet$ with a natural family of isomorphisms $d_{A,B,C} : A \bullet (B \oplus C) \xrightarrow{\sim} A \bullet B \oplus A \bullet C$

We define a *distributive traced category* to be a such a rig category, with a trace with respect to the first product $\oplus$, and the equation

$$\mathbf{Tr}^{X \bullet U}[\mathsf{Id}_X \bullet f] = \mathsf{Id}_X \bullet \mathbf{Tr}^{U}[f] \quad \text{(\textit{modulo} distributivity isomorphisms)}$$

All the categories from the examples are distributive traced categories.

Light Promotion Functors

Definition

In all example we saw, the promotion functor uses the second product.
We define for any object X the light promotion functor $\mathbf{!}_X$:

$$\mathbf{!}_X A = X \bullet A \quad \text{(on objects)}$$
$$\mathbf{!}_X f = \mathsf{Id}_X \bullet f \quad \text{(on morphisms)}$$

Always gives a traced (distributivity/trace equation) functor.

Intuition:

- $\mathbf{!}_X A$ is a type of A-messages, together with an auxiliairy X message.
- $\mathbf{!}_X f$ acts as f on the A part, and simply passes on X messages.

Retractions

Via distributivity, we get for instance

$$!_X A \oplus !_Y A \simeq !_{X \oplus Y} A$$

Moreover, retractions on X lift to natural retractions on $!_X$:

Theorem

For any retraction $f, g : X \lhd Y$ we have a natural retraction $\mathsf{Id}_A \bullet f, \mathsf{Id}_A \bullet g : !_X A \lhd !_Y A$

In particular, if an object has retractions

- $X \oplus X \lhd X$
- $X \bullet X \lhd X$
- $0 \lhd X$
- $1 \lhd X$

then $!_X$ is a promotion functor.

All examples we saw feature a functor of this type.

Representation theorem

A little puzzle: suppose we are in a distributive traced category, under which conditions a promotion functor F is actually implemented (up to isomorphism) as $\mathbf{!}_X$ for some X?

In the examples, we have $FA = X \bullet A$. So in particular $F1 = X \bullet 1 \simeq X$.

So we have $FA = X \bullet A = \mathbf{!}_{F1} A$. We just have to figure out under which conditions this holds:

Theorem

A promotion functor F in a distributive traced category is represented as $\mathbf{!}_X$ if and only if $F(A \bullet B) \simeq FA \bullet B$ for all A, B. and in that case we have $F \simeq \mathbf{!}_{F1}$.

Easy proof: $FA \simeq F(1 \bullet A) \simeq F1 \bullet A = \mathbf{!}_{F1} A$.

Complexity and Retractions?

Provides a semantic point of view on implicit complexity:

- Promotion functors in light logics = source of expressivity (structural rules)
- In the construction we describe, structural rules for $!_X$ depend on the retractions of X. More retractions mean more structural rules.
- Eg. in Schöpp and Dal Lago's `IntML`, finite base types (once bit-width is fixed) and therefore only basic retractions such as $A \lhd A \oplus B$, $1 \lhd \mathtt{int}$.

. . . Thank you!