Proof complexity of deep inference: a survey

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École Normale Supérieure de Lyon

Invited talk at LCC '16

2nd September, 2016
Marseilles

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- There will be *formal* proofs.

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- There will be pictures.
- A deep area with plenty of open problems.

Outline

Proof complexity

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- Proof systems can be compared by polynomial simulation.
- This induces a complexity-theoretic hierarchy of proof systems.
- $\rightsquigarrow$ subproblems of **coNP** vs. **NP**.

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Cut-free LK

Tableaux

Hilbert-Frege

LK

The limits of our knowledge...

Exponential lower bounds

Cut-free LK

Tableaux

No known lower bounds

Hilbert-Frege

LK

The limits of our knowledge...

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Cut-free LK

Tableaux

$\leq$

Resolution

No known lower bounds

Hilbert-Frege

LK

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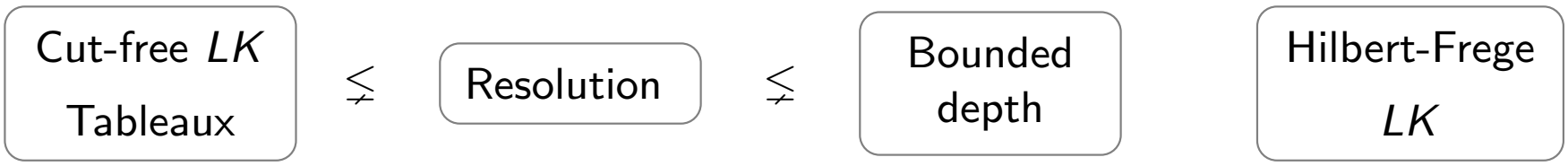

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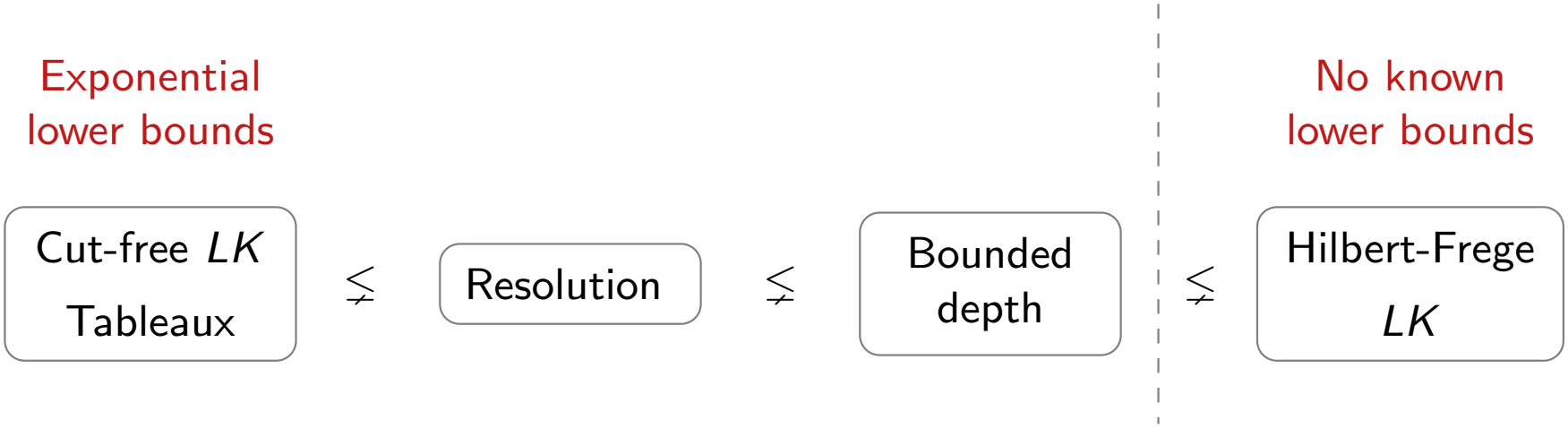

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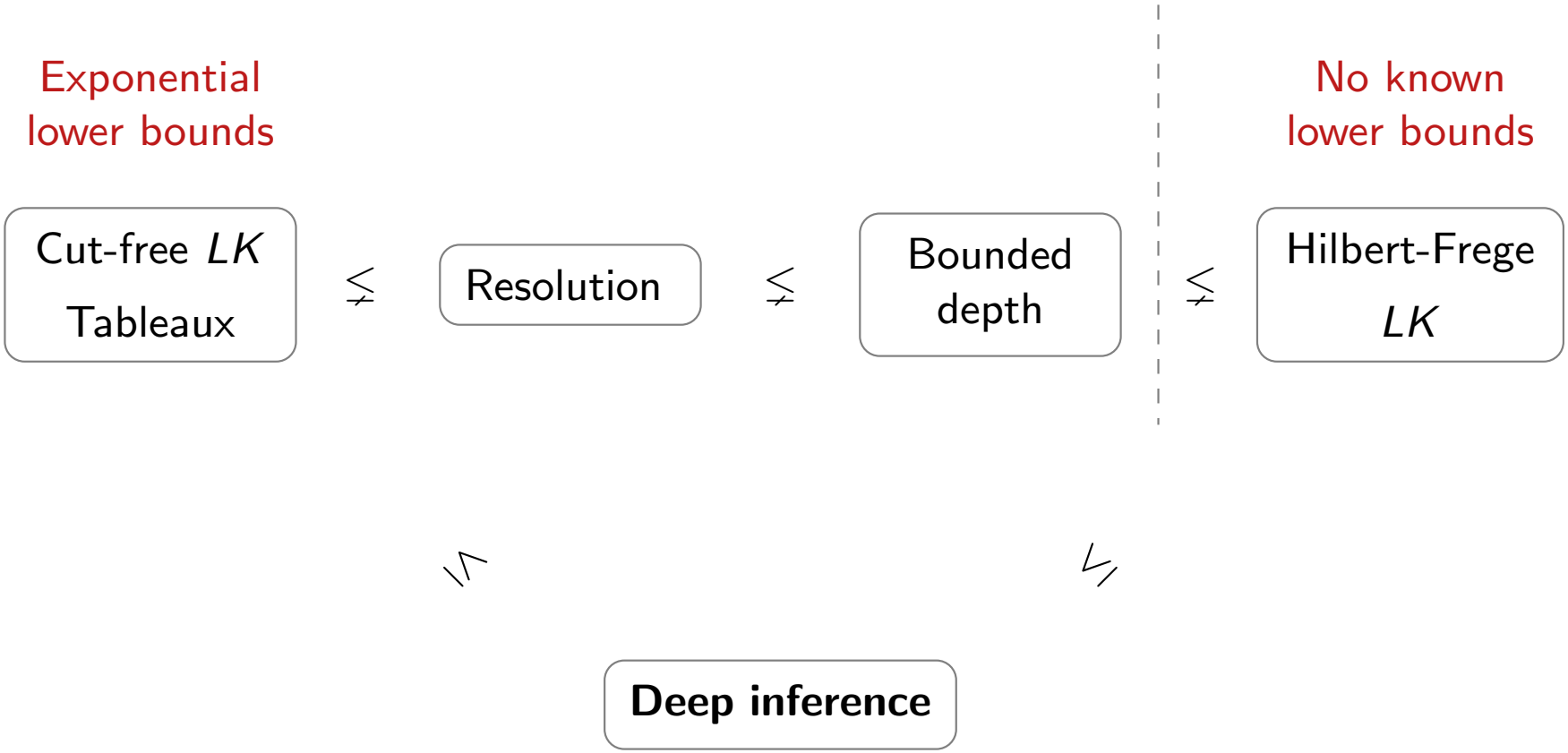

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- Analytic systems for more logics.
- Inference steps can be local.
- Systems have smaller proofs.

Systems for classical propositional logic (Guglielmi, Brünnler & Tiu '01)

The system SKS for propositional logic:

$$\mathsf{i}{\uparrow}\ \frac{A \wedge \bar{A}}{\bot} \qquad \mathsf{w}{\uparrow}\ \frac{A}{\top} \qquad \mathsf{c}{\uparrow}\ \frac{A}{A \wedge A} \qquad \mathsf{s}\ \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

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$$\Phi, \Psi \quad ::= \quad A \quad | \quad \Phi \star \Psi \quad | \quad \rho\,\frac{\Phi}{\Psi}$$

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- Proofs without $\mathsf{i}{\uparrow}$ and $\mathsf{w}{\uparrow}$ are considered analytic.

An example derivation

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{p}{p \wedge p}\, c\uparrow \quad \vee \quad \dfrac{\dfrac{\bot}{p}\, w\uparrow}{p \wedge p}\, c\uparrow
}{p \vee p}\, m
}{p}\, c\downarrow
\quad = \quad
\dfrac{p}{\dfrac{\bot}{q}\, w\downarrow}
}{p \vee p}\, s
\quad \wedge \quad c\downarrow \dfrac{p \vee p}{p}
}{
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- **Slogan:** Deep inference simultaneously optimises checking complexity and communication complexity.

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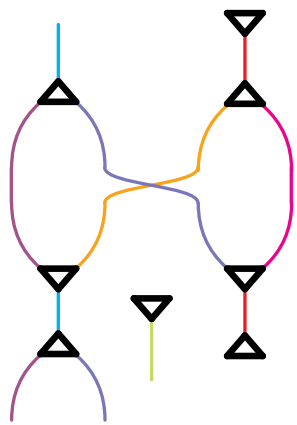

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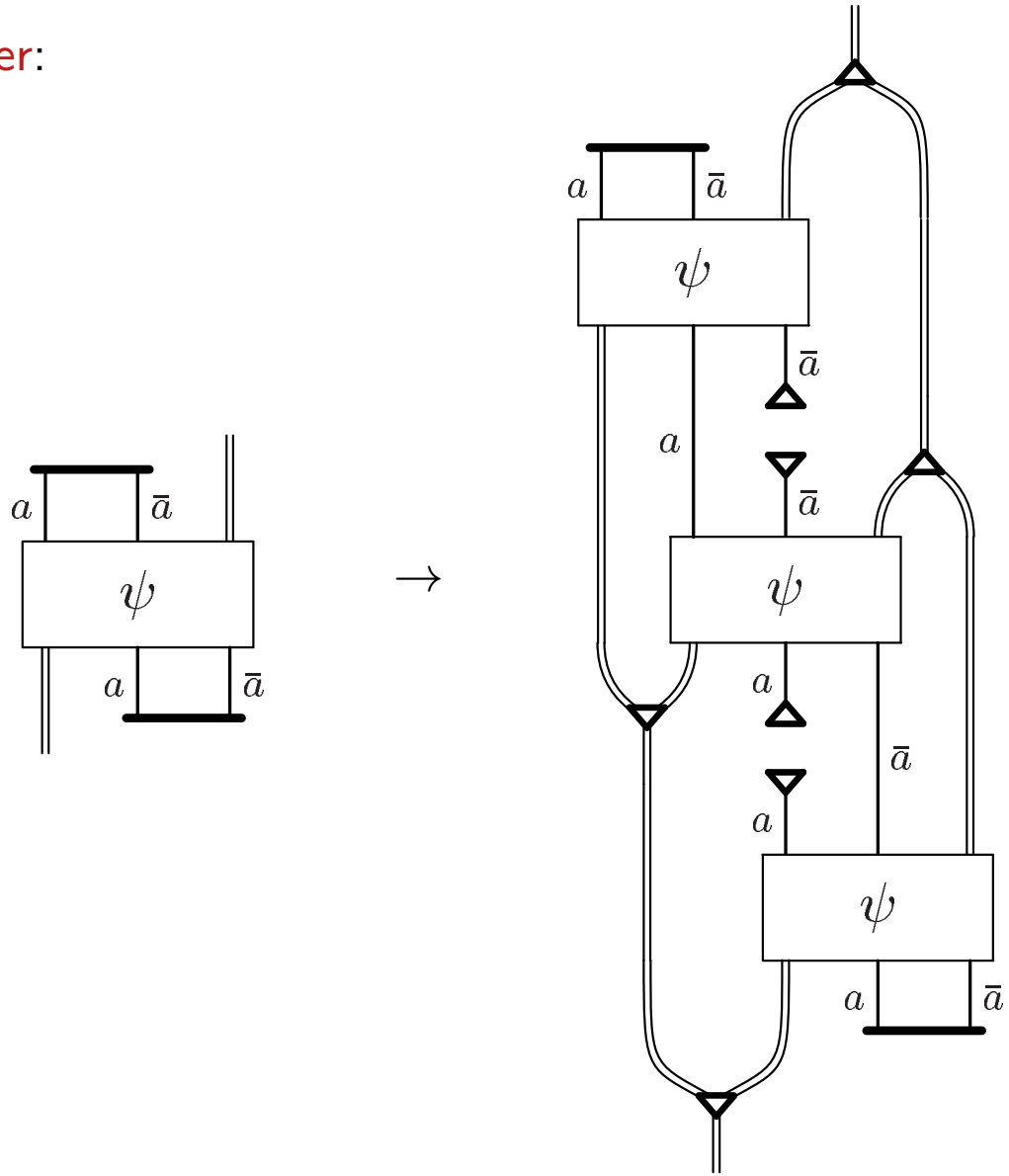
The path breaker:



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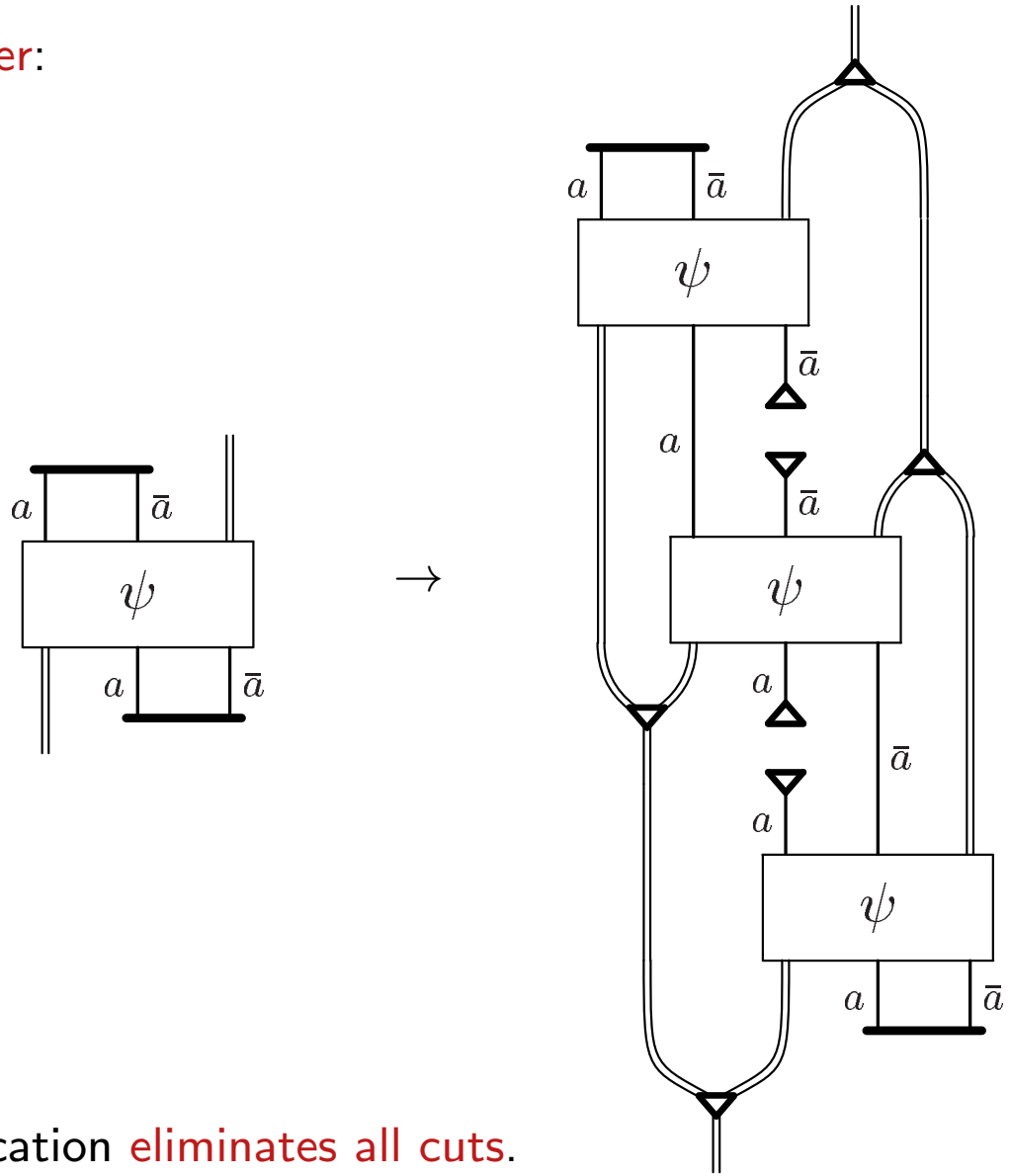

Recursive application eliminates all cuts.

Outline

Doing better than the pathbreaker

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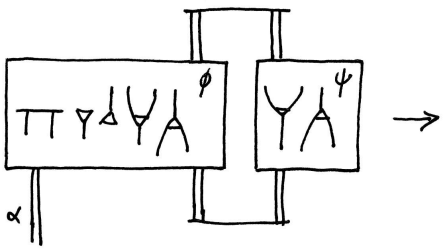
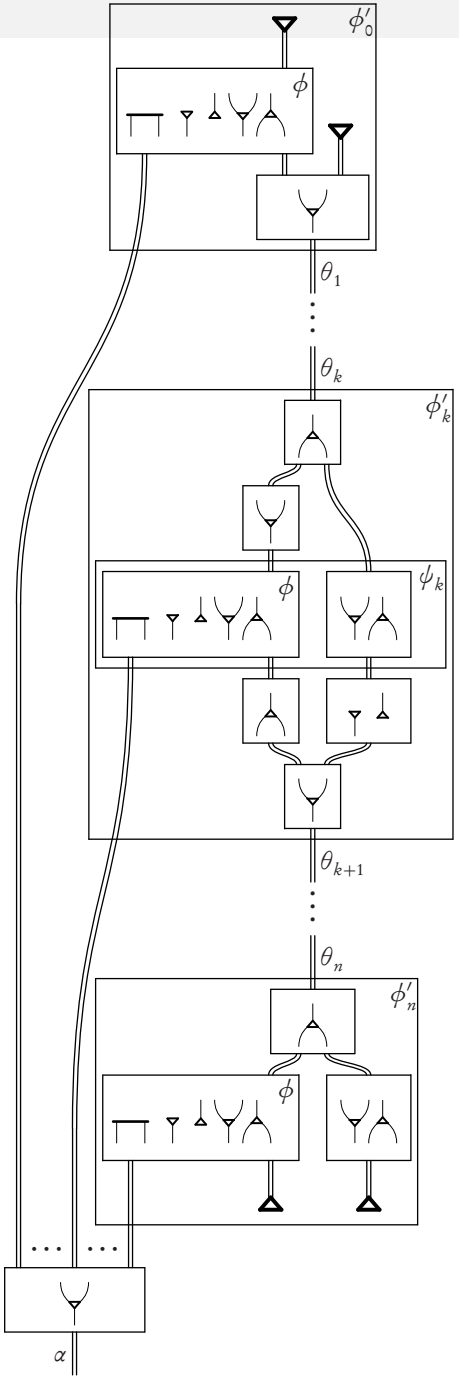
We can obtain a length-width tradeoff:





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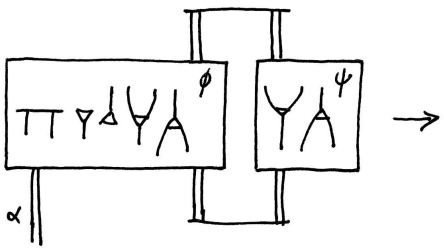

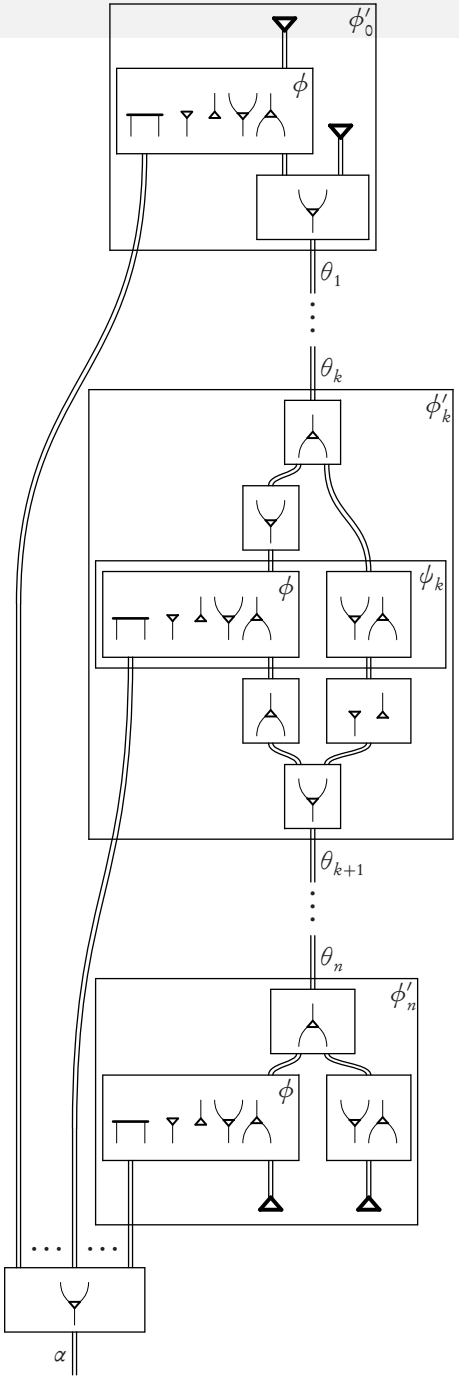

We rely on a logical catalyst...

Formulae for counting

Definition (Threshold functions)

The Boolean functions $TH_k^n : \{0,1\}^n \to \{0,1\}$ are defined by

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- Unfortunately too big: they have size $\approx 2^n$.

Formulae for counting

However, notice the following identity:

$$TH_k^{2n}(\mathbf{a}, \mathbf{b}) \quad = \quad \bigvee_{i=0}^{k} TH_i^n(\mathbf{a}) \wedge TH_{k-i}^n(\mathbf{b})$$

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The following have $n^{O(\log n)}$-size derivations in $\{\mathsf{w}\uparrow, \mathsf{w}\downarrow, \mathsf{s}\}$:

$$\begin{array}{ccc} \top & th_{n+1}^n(\mathbf{a}) & th_k^n(\cdots \bot \cdots) \\ (1)\,\| & (2)\,\| & (3)\,\| \\ th_0^n(\mathbf{a}) & \bot & th_{k+1}^n(\cdots \top \cdots) \end{array}$$

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Proof.

(1) and (2) are simple. (3) requires a bit of work... □

Increasing thresholds: proof of (3)

Proof is by induction on n

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$$= \frac{th_k^{2n}(\cdots \bot \cdots, \boldsymbol{b})}{\bigvee_{i=0}^{k} \left(\begin{array}{c} th_i^n(\cdots \bot \cdots) \\ IH \,\| \\ th_{i+1}^n(\cdots \top \cdots) \end{array} \wedge th_{k-i}^n(\boldsymbol{b}) \right) \vee \mathrm{w}{\downarrow} \dfrac{\bot}{th_0^n(\cdots \top \cdots) \wedge th_{k+1}^n(\boldsymbol{b})}}$$

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(**NB:** Notice the crucial use of deep inference!)

Main result

Lemma

There is a quasipolynomial transformation of the following format,

$$\pi \Big\| \text{SKS} \quad \tau \qquad \longrightarrow \qquad \begin{array}{c} th_k^n(\mathbf{a}) \\ T_k\pi \Big\| \textit{cut-free} \\ \tau \vee th_{k+1}^n(\mathbf{a}) \end{array}$$

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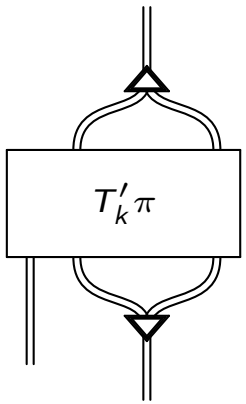

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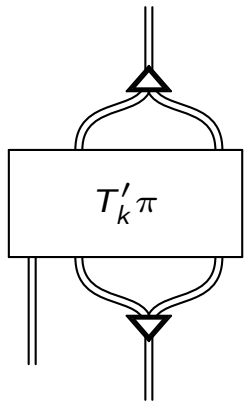

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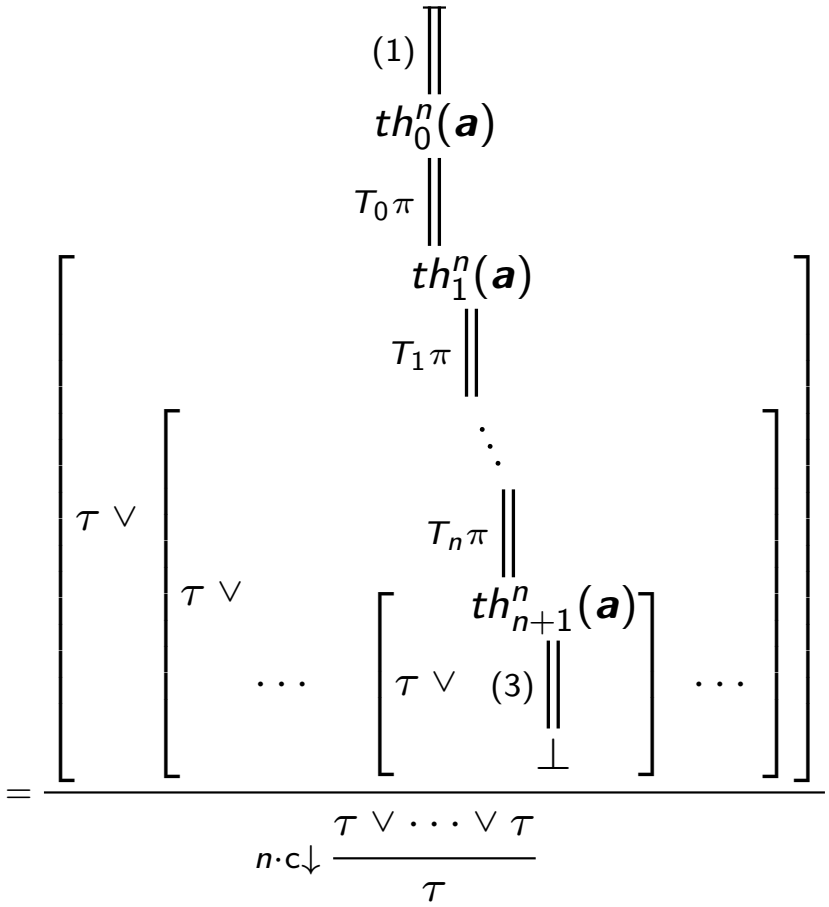
Theorem (Jeřábek '09, Bruscoli et al. '10)

SKS *has quasipolynomial-time cut-elimination.*

Proof of main result



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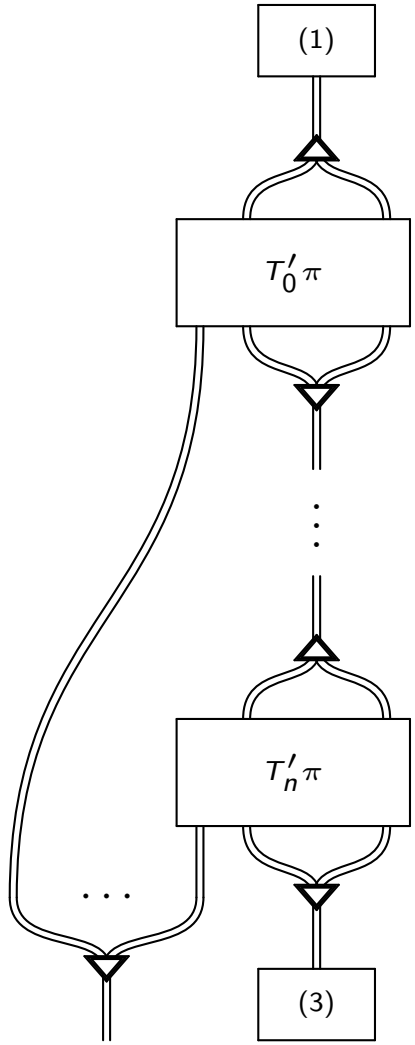
T_0\pi\ \| \\
th_1^n(\boldsymbol{a}) \\
T_1\pi\ \| \\
\vdots \\
T_n\pi\ \| \\
\left[\tau \vee \left[\tau \vee \cdots \vee \left[\tau \vee (3)\ \| \right] \cdots\right]\right]
\end{array}}{n\cdot \mathrm{c}\!\downarrow \dfrac{\tau \vee \cdots \vee \tau}{\tau}}
$$



Outline

Analytic subsystems of deep inference

Recall the system SKS:

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Analytic subsystems of deep inference

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Definition

- $\mathsf{KS}^+ := \mathsf{SKS} \setminus \{\mathsf{i}\uparrow\}$. (cut-free deep inference)
- $\mathsf{KS} := \mathsf{SKS} \setminus \{\mathsf{i}\uparrow, \mathsf{w}\uparrow, \mathsf{c}\uparrow\}$. (minimal deep inference)

From KS^+ to KS

- KS^+ is quite powerful: almost as strong as Hilbert-Frege systems.

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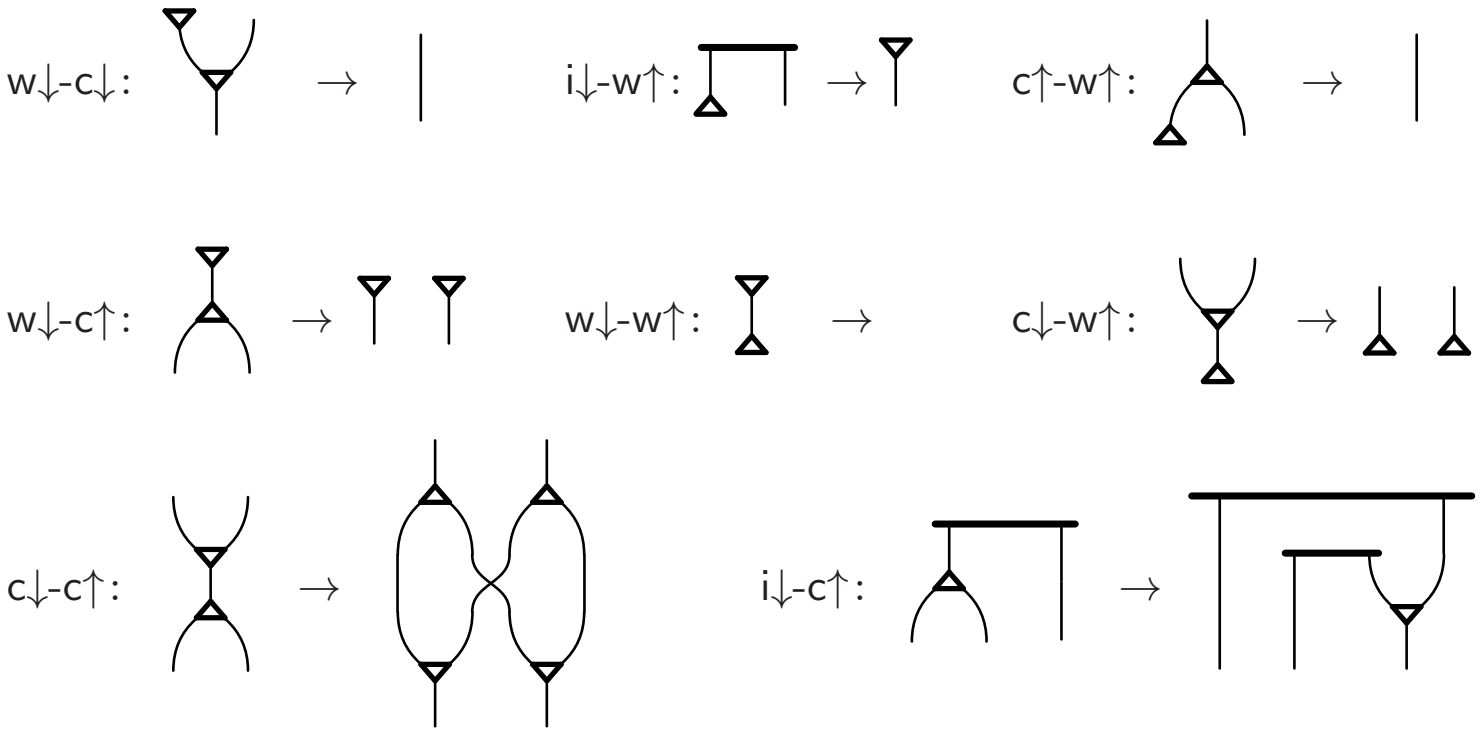
- We consider the question of KS^+ vs. KS via normalisation procedures.
- In particular, we rely on local graph rewriting on atomic flows...

Normalisation procedures

Consider the rewrite system **norm** on atomic flows:



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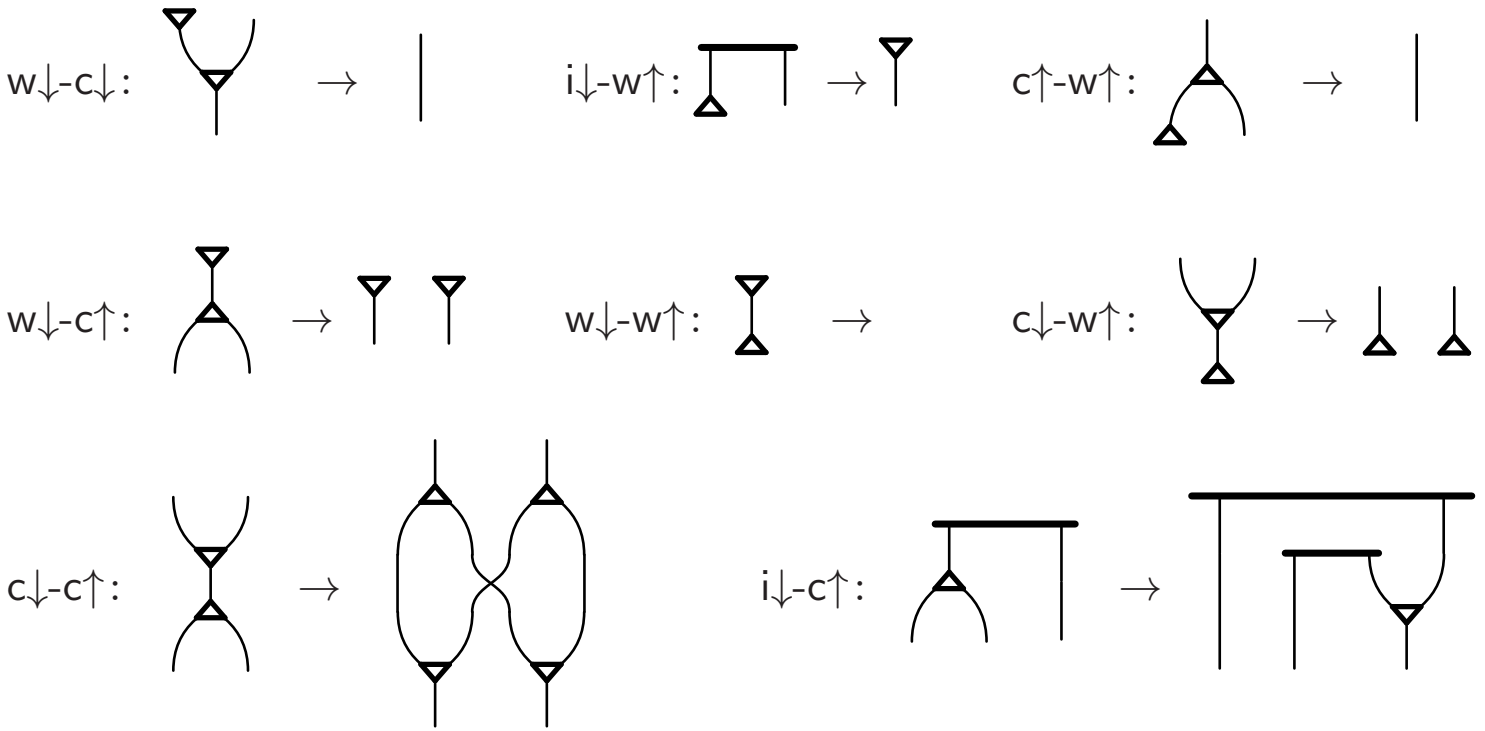

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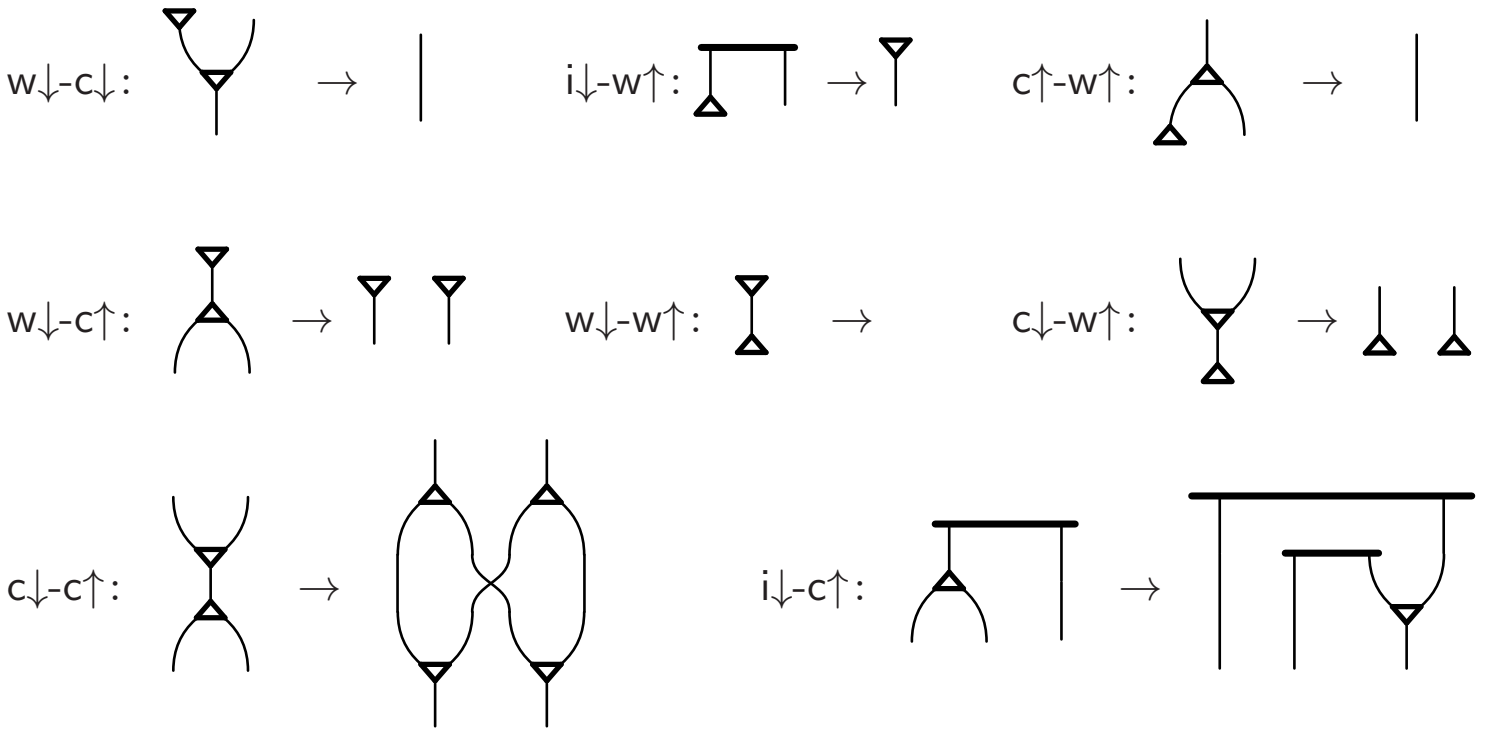

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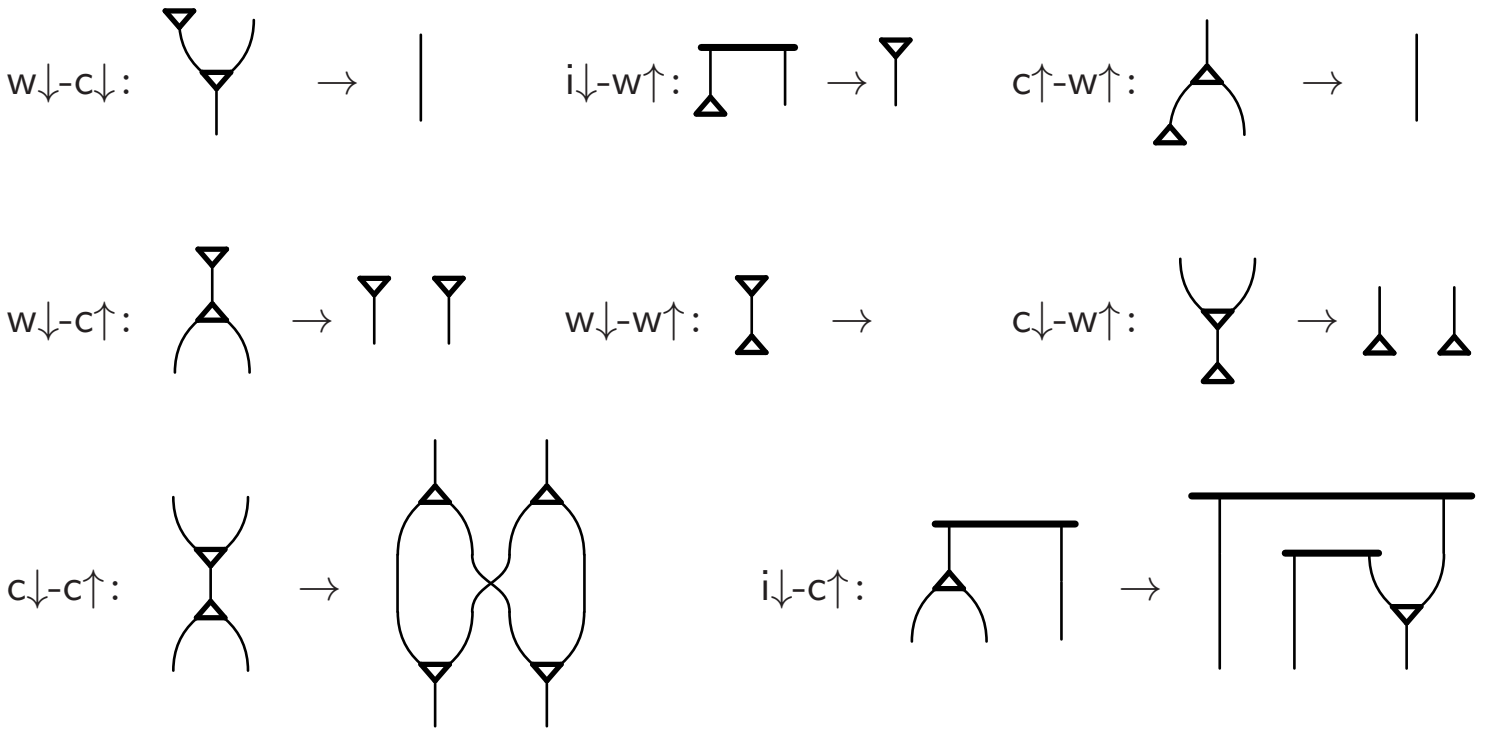
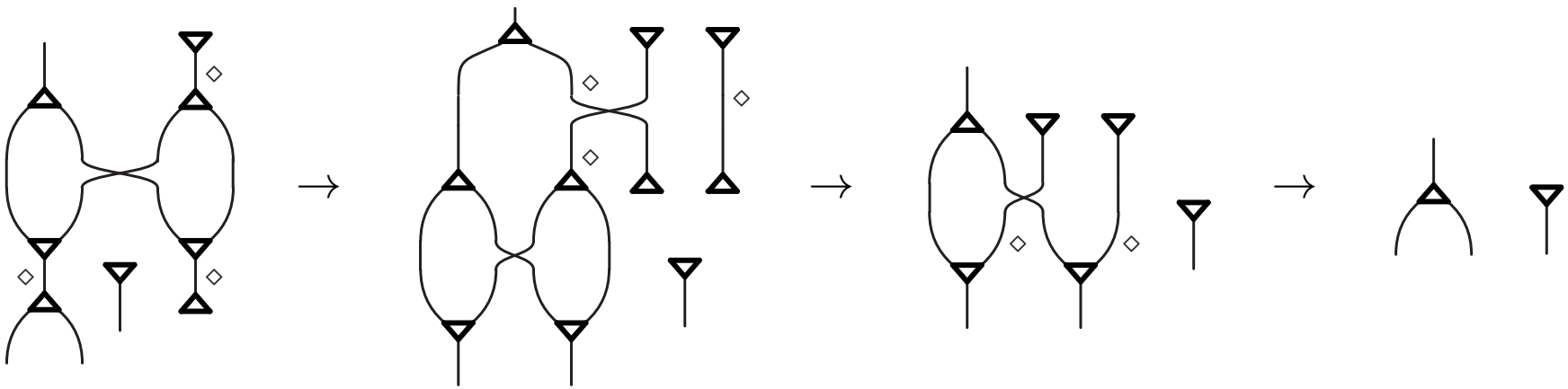


Theorem

1. **norm** *is terminating and confluent.*
2. *Induces normalisation procedure from* KS^+ *to* KS.
3. *Complexity is polynomial time in number of paths of an input flow.*

Example of a flow reduction



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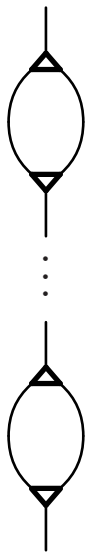

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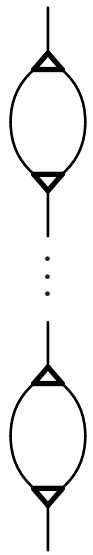

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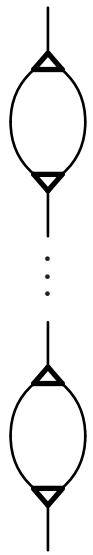

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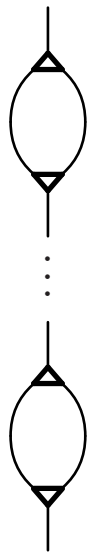

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- **NB:** Quasipolynomial-time normalisation uses flows of length $O(n)$.

Outline

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$$\mathrm{PHP}_n^m := \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} p_{ij} \rightarrow \bigvee_{j=1}^{n} \bigvee_{i \neq i'} (p_{ij} \wedge p_{i'j})$$

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Theorem (D. '14)

There are quasipolynomial-size proofs of PHP_n^{n+1} *in* KS.

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Question

What is the size of proofs of $th_k^n(\mathbf{a}) \to th_k^n(\sigma(\mathbf{a}))$*?*

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- In particular we formulate divide-and-conquer inductions to control the length of atomic flows.

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Observation

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- To achieve the full transposition from these, we need to interleave the rows of these two matrices.

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Lemma

There are KS*-derivations of* $th_k^n(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \to th_k^n(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{d})$ *of size* $n^{O(\log n)}$.

Main results

Theorem

There are KS^+ *proofs of* $th_k^n(\boldsymbol{A}) \to th_k^n(\boldsymbol{A}^\intercal)$ *of size* $n^{O(\log n)}$ *and whose flows have length* $O(\log^2 n)$.

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There are KS *proofs of* PHP_n^m *of size quasipolynomial in* m *and* n.

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From previously known lower bounds, we obtain the following:

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KS *is exponentially separated from:*

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In fact, our proof structure yields a new proof for the weak pigeonhole principle, via threshold approximators:

Theorem (D. '14)

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This is the first time developments in deep inference yielded improved bounds for mainstream proof complexity.

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Dag-like cut-free sequent calculus with elimination rules,

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Open problem

Does d-KS *(quasi)polynomially simulate* KS *for some* d*?*

Relative complexity of KS

How does KS compare with other systems?

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Open problem

KS *vs. bounded-depth Hilbert-Frege systems, in general.*

A uniform approach

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Bounded arithmetic

Propositional systems as nonuniform versions of weak theories of arithmetic.

A uniform approach

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Propositional systems as nonuniform versions of weak theories of arithmetic.

E.g.

Theorem (Paris & Wilkie '81)

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$\rightsquigarrow$ more simulations, superpolynomial lower bounds,... ?

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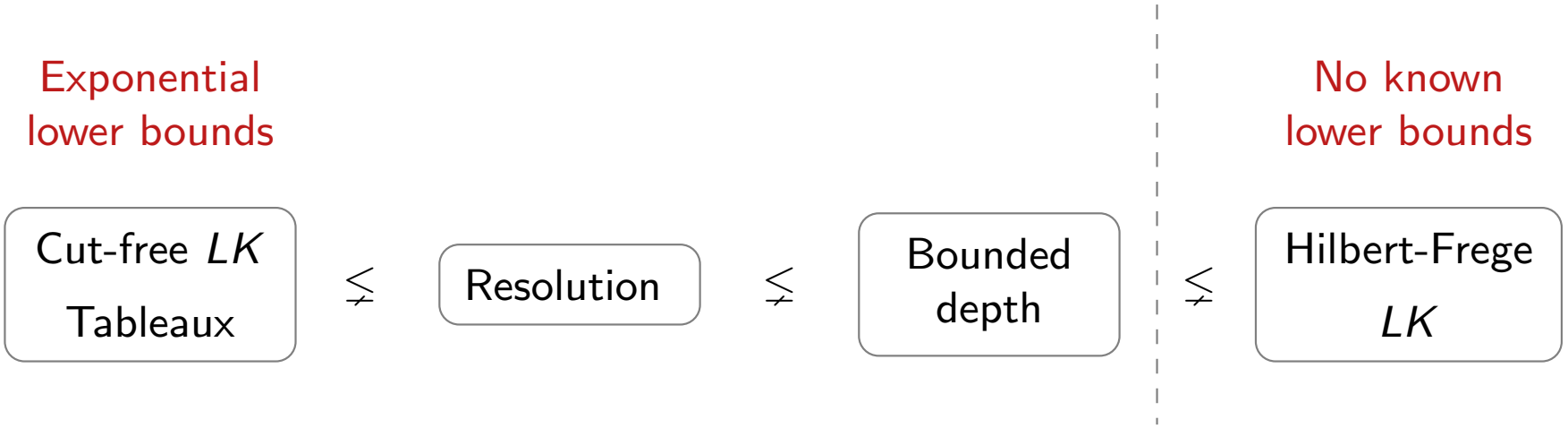
What lies ahead...

- Tighter integration with *structural* proof theory.
- Calibration with powerful tools from combinatorics.

The current situation



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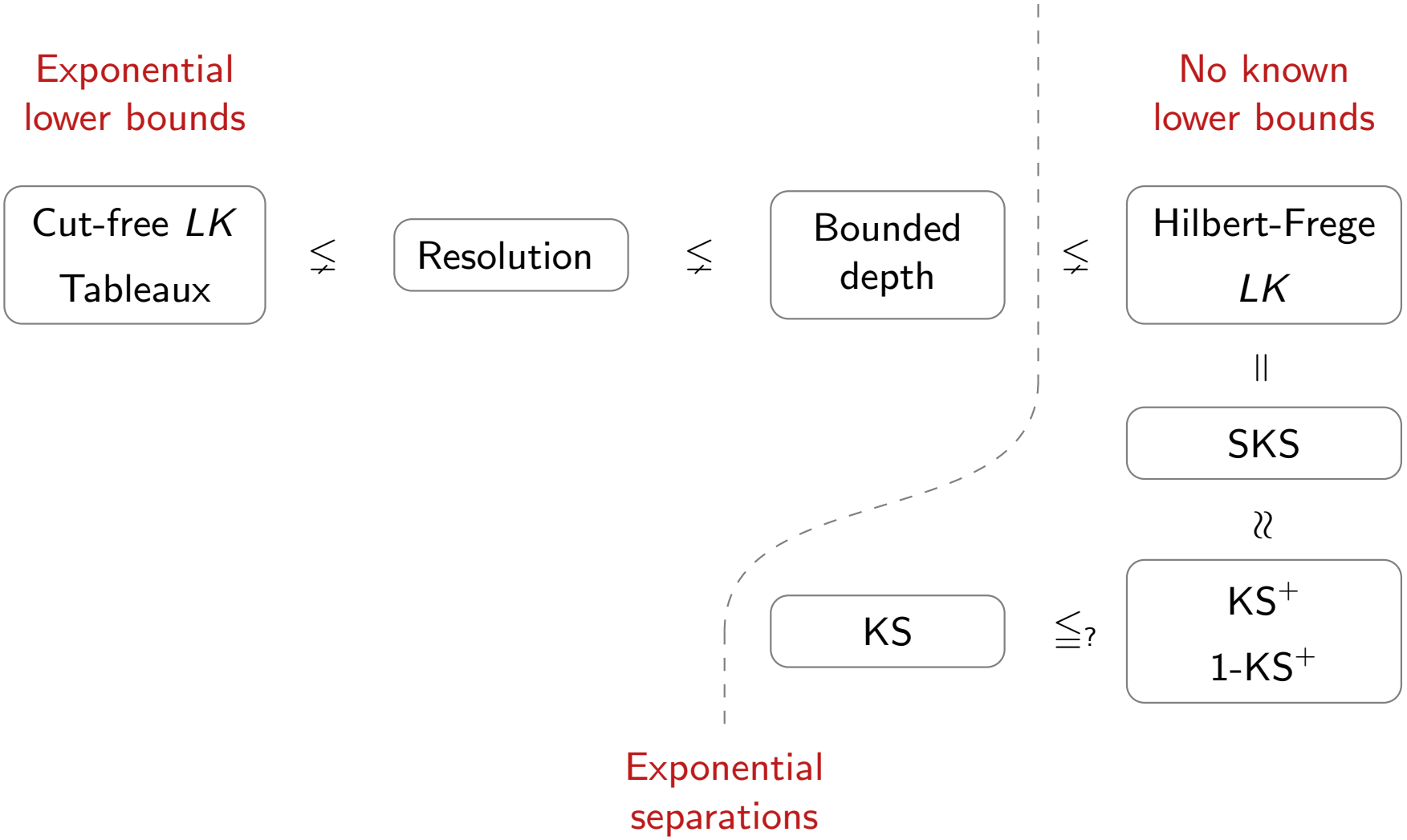

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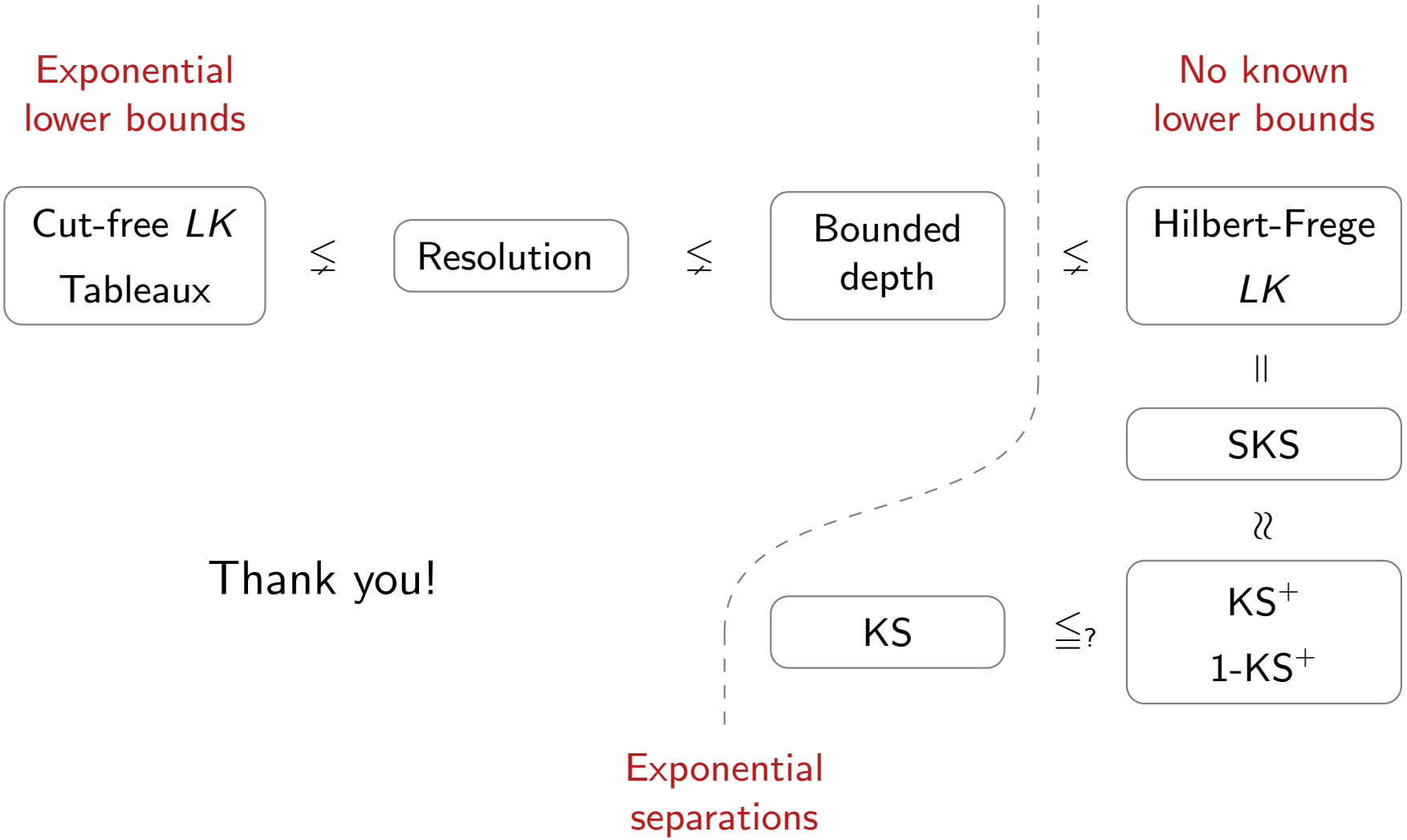

Thank you!