

Infinite domain constraint satisfaction problem

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CSP

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Constraint Satisfaction Problem

$$\mathbf{CSP = CS + L}$$

Fixed-template CSP over **finite** domains

- ▶ **Computer Science**
 - ▶ CSP is a class of computational problems within NP
- ▶ **Logic**
 - ▶ approached via logic, mostly universal algebra

Achievements

- ▶ A lot of tractability/hardness results under one roof
 - such a generality and strength was not dreamt of
- ▶ Better understanding why problems are hard (lack of symmetry)
- ▶ (meta) People from different fields talk
- ▶ Distant areas coming closer
- ▶ Expands: counting, optimization, approximation, **infinite domains**, ...

Fixed-template CSP over **infinite** domains

- ▶ Covers all computational problems (too optimistic to attack now)
- ▶ But there are reasonable restrictions (eg. to be back in NP)
- ▶ Covers way more computational problems in NP than CSP over finite domains
- ▶ More areas of mathematics (mostly logicky) involved
 - ▶ universal algebra
 - ▶ model theory (ω -categorical structures, homogeneity)
 - ▶ Ramsey theory (\leftrightarrow abstract topological dynamics)
- ▶ More people talk to each other, etc

- ▶ Some general propaganda (it's over)
- ▶ CSP and examples
- ▶ Complexity classification conjectures
- ▶ News

Disclaimer

- ▶ No pictures
- ▶ Most of the definitions imprecise
- ▶ Almost no theorem true as stated

CSP and examples

Fixed-template CSP

Fix $\mathbb{A} = (A; R_1, R_2, \dots, R_n)$: relational structure (each $R_i \subset A^k$)

Definition

Instance of CSP(\mathbb{A}): Expression of the form

$$R_1(x, y, z), R_2(t, z), R_1(y, y, z), \dots$$

where each R_i is in \mathbb{A} .

Question: Is there a map: variables $\rightarrow A$ satisfying all constraints?

- ▶ Other variants: nothing is fixed, something else is fixed (includes industrial problems)
- ▶ Even within fixed-template, many other questions, eg.
 - ▶ Count the number of solutions
 - ▶ Optimize the number of satisfied constraints
 - ▶ Approximately optimize the number of satisfied constraints
- ▶ This talk: only the above version, only computational complexity

Boolean satisfiability problems, (di)graph coloring problems, solving systems of equations over finite domain

- ▶ 3-SAT: $\mathbb{A} = (\{0, 1\}; x \vee y \vee z, x \vee y \vee \neg z, \dots)$
- ▶ 3-COLOR: $\mathbb{A} = (\{0, 1, 2\}; x \neq y)$
- ▶ q -LIN: $\mathbb{A} = (GF(q); \text{affine subspaces})$

Problems in temporal and spatial reasoning, and many more AI areas, computational linguistics, phylogenetic reconstruction (see [Bodirsky'12](#))

- ▶ ACYCLICITY: $\mathbb{A} = (\mathbb{Q}; x < y)$
- ▶ BETWEENNESS: $\mathbb{A} = (\mathbb{Q}; y \text{ between } x \text{ and } z)$
- ▶ GRAPH-SAT(F), where F is a set of finite graph formulas
INPUT: $\Phi_1(\text{variables}), \Phi_2(\text{variables}), \dots$ with $\Phi_i \in F$
OUTPUT: Is it satisfiable in a graph?

Complexity classification conjectures

- ▶ **Crucial:** If \mathbb{A} pp-interprets \mathbb{B} , then $\text{CSP}(\mathbb{B})$ is easier than $\text{CSP}(\mathbb{A})$
 - ▶ \mathbb{A} **pp-defines** \mathbb{B} = the same domain, and relations in \mathbb{B} definable using relations in \mathbb{A} , and $\exists, =, \wedge$.
 - ▶ \mathbb{A} **pp-interprets** \mathbb{B} if the domain of \mathbb{B} is a pp-definable subset of A^k modulo a pp-definable equivalence and all the relations of \mathbb{B} are “pp-definable” from \mathbb{A}
- ▶ **Corollary:** If \mathbb{A} pp-interprets some structure with NP-hard CSP (like 3-SAT), then $\text{CSP}(\mathbb{A})$ is NP-hard
- ▶ **Remark:** \mathbb{A} pp-interprets 3-SAT, then it pp-interprets every finite structure
- ▶ **Tractability conjectures:** If \mathbb{A} is (*) and does not interpret 3-SAT then $\text{CSP}(\mathbb{A})$ is in P
 - ▶ (*) finite [Feder,Vardi'93],[Bulatov,Jeavons,Krokhin'00]
 - ▶ (*) reduct of a countable finitely bounded homogeneous structure [Bodirsky,Pinsker'11]

More on tractability conjecture
for **finite** structures

Polymorphism clone

- ▶ **Def:** $\text{Pol}(\mathbb{A}) = \{f \mid f : \mathbb{A}^n \rightarrow \mathbb{A} \text{ homo}\}$ the set of **polymorphisms**
 - ▶ Polymorphisms generalize endomorphisms (automorphisms)
 - ▶ Polymorphisms are symmetries of higher arity
- ▶ **Fact:** $\text{Pol}(\mathbb{A})$ is a **clone**: contains projections, closed under composition
- ▶ **Def:** **Homomorphism** between clones $\mathcal{A} \rightarrow \mathcal{B}$: mapping from operations in \mathcal{A} to operations in \mathcal{B} that
 - ▶ preserves projections and composition
 - ▶ equivalently, preserves (universally quantified) equations;
eg. an associative/commutative operation is mapped to an associative/commutative operation

- ▶ **Thm:** Finite \mathbb{A} pp-interprets finite \mathbb{B}
iff \exists homo $\text{Pol}(\mathbb{A}) \rightarrow \text{Pol}(\mathbb{B})$
 - ▶ pp-definitions \leftrightarrow polymorphisms [Geiger'68, Bodnarčuk, Kalužnin, Kotov, Romov'69]
 - ▶ pp-interpretations \leftrightarrow standard algebraic constructions (powers, subalgebras, quotients) [Bodirsky, Willard]
 - ▶ standard constructions \leftrightarrow equations (clone homomorphisms) [Birkhoff'35]
- ▶ **Trivial clone \mathcal{P} :** clone of projections on (say) 2-element set
 - ▶ a.k.a. 0, 1, 2
 - ▶ **Example:** $\text{Pol}(3\text{-SAT})$.
- ▶ **Corollary:** Finite \mathbb{A} pp-interprets all finite
iff \exists homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$

ie. the set of equations satisfied by polymorphisms of \mathbb{A}
is satisfiable by projections

Tractability conjecture

Conjecture (Bulatov, Jeavons, Krokhin'00)

Assume \mathbb{A} finite.

If \exists homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$, then $\text{CSP}(\mathbb{A})$ in \mathcal{P} .

Recall: Otherwise $\text{CSP}(\mathbb{A})$ is NP-complete.

Theorem

Assume \mathbb{A} finite, TFAE

- ▶ \mathbb{A} does not pp-interpret all finite
- ▶ \exists homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$
ie. polymorphisms satisfy nontrivial equations
- ▶ $\text{Pol}(\mathbb{A})$ contains an operation s of arity 6 such that
 $s(x, y, x, z, y, z) = s(y, x, z, x, z, y)$ [Siggers'10]

3rd item: concrete and positive alternative

Conjecture (Bulatov, Jeavons, Krokhin; Siggers)

Assume \mathbb{A} finite. TFAE (assume $P \neq NP$)

- ▶ $\text{CSP}(\mathbb{A})$ in P
- ▶ $\text{Pol}(\mathbb{A})$ contains an operation s of arity 6 such that

$$s(x, y, x, z, y, z) = s(y, x, z, x, z, y)$$

Reality can be worse:

Finite tractability conjecture vs. reality

Future theorem (T.B.D. 2048)

Assume \mathbb{A} finite. TFAE

- ▶ $\text{CSP}(\mathbb{A})$ in P
- ▶ $\text{Pol}(\mathbb{A})$ contains operations t_1, t_2, \dots such that

$$t_1(x_3, x_1, x_2, x_2) = t_2(x_3, x_3, x_1)$$

$$t_{20}(x_1, x_1, x_2) = t_{13}(x_2, x_1)$$

...

...but certainly the characterization looks somewhat like this
[Barto, Pinsker' ?]

ie. depends only on height 1 equations

(could be a disjunction of such conditions, at worst)

More on tractability conjecture
for **infinite** structures

ω -categorical structures

- ▶ Every computational problem is P-time equivalent to $\text{CSP}(\mathbb{A})$ [Bodirsky, Grohe'08]
- ▶ Some parts of the theory work well for countable ω -categorical structures
- ▶ Countable \mathbb{A} is ω -categorical if
 - ▶ it is the unique countable model of its first order theory
 - ▶ equivalently, action of $\text{Aut}(\mathbb{A})$ on n -tuples has finitely many orbits $\forall n$
- ▶ **Examples:** $(\mathbb{Q}, <)$, random graph, random poset
- ▶ CSPs over ω -categorical structures still essentially cover all computational problems (!) [Bodirsky, Grohe'08]

A reasonable class

- ▶ The Pinsker–Bodirsky infinite tractability conjecture concerns **reducts of finitely bounded homogeneous structures**
 - ▶ **homogeneous** = isomorphisms between finite induced substructures extend to automorphisms
 - ▶ **finitely bounded** = finite induced substructures can be characterized by finitely many forbidden substructures
 - ▶ **reduct** = has a first order definition in it
- ▶ they are ω -categorical
- ▶ their CSP is in NP
- ▶ **Examples:** $(\mathbb{Q}, <)$, random graph, random poset and reducts
- ▶ Complexity classified for these examples
[Bodirsky, Kára'09], [Bodirsky, Pinsker'15], [Kompatscher, Van Pham]

pp-interpretations and clone homomorphisms

- ▶ $\text{Pol}(\mathbb{A})$ carries a natural topology of pointwise convergence (metrizable)
- ▶ **Thm:** ω -categorical \mathbb{A} pp-interprets finite \mathbb{B}
iff \exists **continuous** homo $\text{Pol}(\mathbb{A}) \rightarrow \text{Pol}(\mathbb{B})$
 - ▶ pp-definitions \leftrightarrow polymorphisms [Bodirsky, Nešetřil'06]
 - ▶ pp-interpretations \leftrightarrow **finite** powers, subalgebras, quotients
 - ▶ these constructions \leftrightarrow continuous clone homomorphisms [Bodirsky, Pinsker'15]
- ▶ **Corollary:** ω -categorical \mathbb{A} pp-interprets all finite
iff \exists continuous homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$
 - ▶ how to translate it to human language?

Tractability conjecture

Conjecture (Bodirsky, Pinsker'11)

Assume \mathbb{A} reduct of finitely bounded homogeneous.
If \exists continuous homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$, then $\text{CSP}(\mathbb{A})$ in P .

Recall: Otherwise $\text{CSP}(\mathbb{A})$ is NP-complete.

Theorem

Assume \mathbb{A} reduct of \dots TFAE

- ▶ \mathbb{A} does not pp-interpret all finite
- ▶ \exists continuous homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$

Questions:

- ▶ is continuity important? (Why do we care?)
- ▶ is there a concrete positive alternative?

“Proof” of the conjecture

Conjecture (Bodirsky, Pinsker'11)

Assume \mathbb{A} reduct of finitely bounded homogeneous.

If \exists continuous homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$, then $\text{CSP}(\mathbb{A})$ in \mathcal{P} .

“Proof”.

Assume \exists continuous homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$

- (1) \exists homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$ ie. \mathbb{A} has nontrivial polymorphisms
- (2) WLOG \mathbb{A} is a reduct of a homogeneous **Ramsey** structure
- (3) From Ramsey \Rightarrow some associated finite structure has non-trivial polymorphisms
- (4) Use the algorithm from finite CSP and
- (5) lift it to the original CSP



Gaps in all 5 stages! Progress on (5) by [Bodirsky and Mottet](#)

News (all positive)

- ▶ **Topology is irrelevant + positive alternative**

Read: Barto, Pinsker: The algebraic dichotomy conjecture for infinite domain CSP, LICS'16

- ▶ **We have a better conjecture**

Read: Barto, Opršal, Pinsker: The wonderland of reflections

- ▶ **We do not have a better conjecture**

Read: Olšák: ?

(1) is done

Theorem (Barto, Pinsker'16)

Assume \mathbb{A} ω -categorical (includes reducts of ...). TFAE

- ▶ \mathbb{A} does not pp-interpret all finite
- ▶ \exists continuous homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$
- ▶ \exists homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$
- ▶ $\text{Pol}(\mathbb{A})$ contains unary α, β and 6-ary s

$$\alpha(s(x, y, x, z, y, z)) = \beta(s(y, x, z, x, z, y))$$

Proof: Finite + compactness + extras

But: We can't claim that the complexity for sure only depends on equations

New conjecture (Barto, Opršal, Pinsker)

Assume \mathbb{A} reduct of finitely bounded homogeneous.

If \exists **uniformly continuous height 1** homo $\text{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$, then $\text{CSP}(\mathbb{A})$ in P .

- ▶ Stronger topological requirement
- ▶ **But:** continuous homo exists iff uniformly continuous exists [Bodirsky, Pinsker'15]
- ▶ Weaker algebraic requirement (only height 1 equations)
- ▶ **Fact:** Otherwise $\text{CSP}(\mathbb{A})$ is NP-complete.
- ▶ **So:** Old conjecture true \Rightarrow new conjecture true

The two conjectures are equivalent

Theorem (Olśák)

The two conjectures are equivalent

- ▶ **Question:** Can we drop the (uniform) continuity from the new conjecture?

> 20% done :)

Join the party until it's late!

Thank you!

Proof of pseudo-Siggers via a pseudo-loop lemma

Loop lemmata

Loop lemma: Let $R \subseteq B^2$, B finite.

Then R contains a loop (a, a) provided

- ▶ certain structural assumption is satisfied, like
 - (1) R is symmetric and contains a triangle
 - (2) R is symmetric and contains an odd cycle
 - (3) R is strongly connected, GCD of cycles lengths = 1
 - (4) R has no sources or sinks and has algebraic length 1
- ▶ and (B, R) does not pp-interpret, with parameters, \mathbb{K}_3

Each loop lemma gives equations [Siggers'10] using a standard universal algebraic argument [Kearnes, Marković, McKenzie'14]

Loop lemma proved by

- ▶ \sim [Hell, Nešetřil'90] assuming (2): purely relational proof
- ▶ [Bulatov'05] assuming (2): relations + operations
- ▶ [Barto, Kozik, Niven'09] assuming (4): purely algebraic proof

Theorem (Siggers'10)

TFAE for finite core \mathbb{A}

- (1) \mathbb{A} does not pp-interpret \mathbb{K}_3 (with parameters).
- (2) \mathbb{A} has a polymorphism s satisfying $s(x, y, x, z, y, z) = s(y, x, z, x, z, y)$.

Only (1) \Rightarrow (2) interesting.

- ▶ Define $B = A^{A^3}$ (element is a 3-ary operation on A)
- ▶ Define

$$R = \left\{ \left(\begin{array}{l} (x, y, z) \mapsto s(x, y, x, z, y, z) \\ (x, y, z) \mapsto s(y, x, z, x, z, y) \end{array} \right) \mid s \text{ a 6-ary polymorphism} \right\}$$

- ▶ Observe
 - ▶ R is symmetric
 - ▶ R contains a triangle (x, y, z form a triangle)
 - ▶ \mathbb{A} pp-interprets $(B; R)$, thus $(B; R)$ does not interpret \mathbb{K}_3
- ▶ Loop in R gives the Siggers operation

To the infinity

Two issues with generalization to ω -categorical:

- ▶ A^{A^3} is not a finite power
 - ▶ Use A^X for X finite subsets of A^3
 - ▶ This gives “locally nice operations”
 - ▶ Compactness argument using ω -categoricity \rightarrow globally nice operations
- ▶ loop lemma does not hold (eg. $(\mathbb{N}; \neq)$)
 - ▶ loop lemma \rightarrow pseudo-loop lemma
 - ▶ pseudo-loop lemma: the technical core

We get:

Theorem

Let \mathbb{A} be a core ω -categorical structure. TFAE

- (1) \mathbb{A} does not pp-interpret \mathbb{K}_3 (with parameters).
- (2) \mathbb{A} has polymorphisms α, β, s satisfying $\alpha s(x, y, x, z, y, z) = \beta s(y, x, z, x, z, y)$.

Theorem

Let $R \subseteq B^2$, B countable.

Let a group G acts on B oligomorphically, let R be G -invariant. Then R contains a pseudo-loop (a, b) , a, b in the same G -orbit provided

- ▶ R is symmetric and contains a triangle
 - ▶ and $(B; R, G\text{-orbits of pairs})$ does not pp-interpret, with parameters, \mathbb{K}_3
-
- ▶ First attempt: use B,K,N algebraic approach. Not successful yet, but **very** interesting “side product” – Olšák’s equations
 - ▶ Second attempt: use Bulatov’s relational/algebraic approach. Success, proof requires generalizations and extra work