Bifibrational Parametricity

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&

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Talk outline

1. **Background**
   System F and $\lambda 2$-fibrations.

2. **Parametricity**
   Parametricity and Reynolds' relational interpretation.

3. **Our model**
   Bifibrational functorial semantics of parametric polymorphism.

4. **Universal parametricity**
   Universal property for the interpretation of forall types.

Part I

System F

and

$\lambda 2$-fibrations

Polymorphic functions

Polymorphic functions: depending on type variables $f : \forall X.T(X)$.

Examples:

- the term

  $$++ : \forall X.X \to X \to X$$

  sum natural number, concatenation lists, ...

- the term

  $$\text{rev} : \forall X.\mathit{list}(X) \to \mathit{list}(X)$$

  reverses lists.

System F

Type context: $\Gamma = X_1, \ldots, X_n$ list of type variables.

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Type judgments:

$$\frac{X_i \in \Gamma}{\Gamma \vdash X_i \text{ type}}$$

$$\frac{\Gamma \vdash T \text{ type} \quad \Gamma \vdash U \text{ type}}{\Gamma \vdash T \to U \text{ type}} \qquad \frac{\Gamma, X \vdash T \text{ type}}{\Gamma \vdash \forall X.T \text{ type}}$$

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Context: Γ, Δ such that $\Gamma \vdash T_i$ type for every $i \in \{1, \ldots, m\}$.

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Term judgments for polymorphic functions:

$$\frac{\Gamma, X, \Delta \vdash t \colon T}{\Gamma, \Delta \vdash \Lambda X . t \colon \forall X . T} \qquad \frac{\Gamma, \Delta \vdash t \colon \forall X . T \quad \Gamma \vdash A \ \text{type}}{\Gamma, \Delta \vdash t\, A \colon T[A]}$$

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Categorical model: $\lambda 2$-fibrations

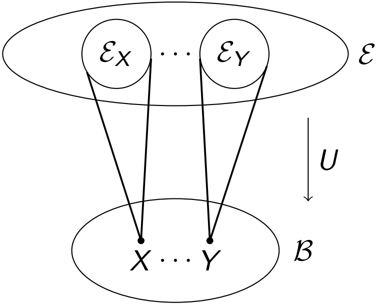
Fibres

Let $U\colon \mathcal{E} \to \mathcal{B}$ be a functor

Definition (fibre)

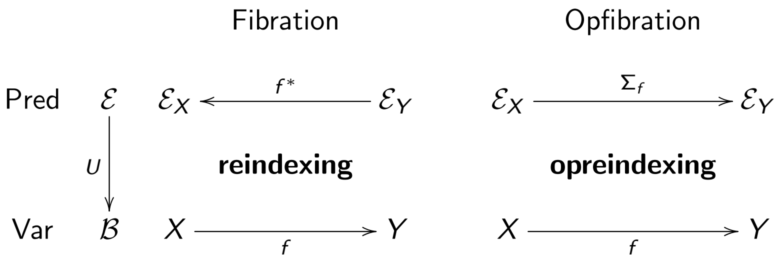
Given an object $X \in \mathcal{B}$, the **fibre** *over X, denoted $\mathcal{E}_X$, is the subcategory of $\mathcal{E}$ consisting of*

- *objects: $A \in \mathcal{E}$ such that $U(A) = X$;*
- *morphisms: $f\colon A \to B$ such that $U(f) = id_X$.*



(Op)fibrations

Transport predicates along change of variables.



They satisfy a universal property.

$\mathcal{E}$ = total category

$\mathcal{B}$ = base category

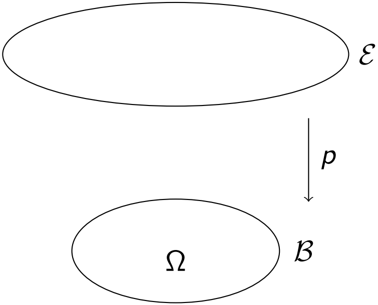
Fibration + opfibration = bifibration.

λ2-fibration 1: type context

Fibration $p\colon \mathcal{E} \to \mathcal{B}$.

Generic object Ω: it represents type variables.

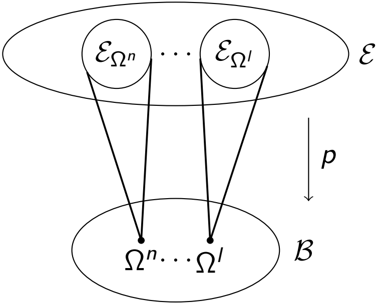
$\mathcal{B}$ has products: Ω^n is the interpretation of $\Gamma = X_1, \ldots, X_n$.



$\lambda 2$-fibration 2: types and terms

Let $\Gamma \vdash T$ type be a type judgment with $\Gamma = X_1, \ldots, X_n$.

We interpret a type T as an object $[\![T]\!]$ in the fibre $\mathcal{E}_{\Omega^n}$.



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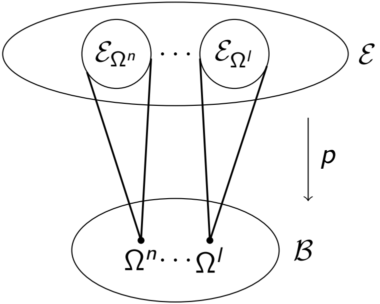
Fibres cartesian closed: products and functions spaces.



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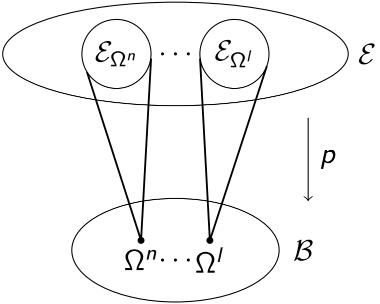
Product: $\Delta = t_1 \colon T_1, \ldots, t_m \colon T_m$ interpreted as $[\![T_1]\!] \times \ldots \times [\![T_m]\!]$.



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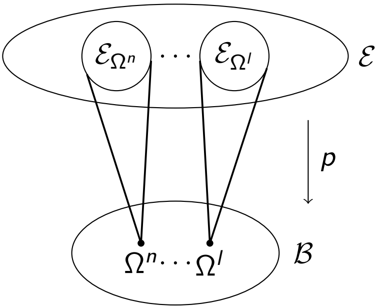
Function space: $\Gamma, \Delta \vdash U \to V$ type interpreted as $[\![U]\!] \Rightarrow [\![V]\!]$.



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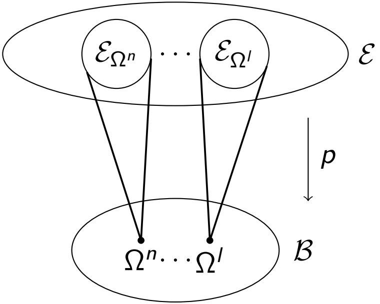
Terms are interpreted as morphisms in the fibres.



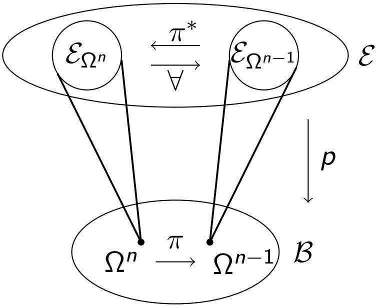
λ2-fibration 3: forall types

We use the adjunction

$$\pi^* \dashv \forall$$

where $\forall$ is right adjoint to the reindexing π^* along the projection

$$\pi \colon \Omega^n \to \Omega^{n-1}$$



We say that p has **simply products** (Beck-Chevaley condition).

Part II

Parametricity

and

Reynolds' relational interpretation

Parametricity

- **Ad hoc polymorphism**
  Defined in a different way depending on the type.
  Example:
  $$++ : \forall X.X \to X \to X$$
  sum natural number $\neq$ concatenation lists.

- **Parametric polymorphism**
  Defined in the same way for any type.
  Example:
  $$\mathrm{rev} : \forall X.\mathit{list}(X) \to \mathit{list}(X)$$

Usefulness of parametricity

- Extract properties from types.

Example: every parametric function

$$h\colon \forall X.\mathit{list}(X) \to \mathit{list}(X)$$

satisfies

$$h\,(\mathrm{map}\, f\ xs) = \mathrm{map}\, f\ (h\, xs).$$

Thanks to Reynolds' relational interpretation.

Reynolds' relational model of System F

- **Relations**
  - Relations are subsets $R \subseteq A \times B$ (Rel).
  - Equality $\mathrm{Eq}\colon \mathsf{Set} \to \mathsf{Rel}$ plays special role.

- **Relational semantics for types**
  - $[\![T]\!]_0\colon |\mathsf{Set}|^n \to \mathsf{Set}$
  - $[\![T]\!]_1\colon |\mathsf{Rel}|^n \to \mathsf{Rel}$

- **Relational semantics for terms**
  - Natural transformation $[\![t]\!]_0$ between functors $|\mathsf{Set}|^n \to \mathsf{Set}$
  - Natural transformation $[\![t]\!]_1$ between functors $|\mathsf{Rel}|^n \to \mathsf{Rel}$

This model has sense only considering impredicativity in CoC, or in the (intuitionistic) internal language of a topos.

Reynolds' key theorems for parametricity

- **Identity Extension Lemma** (IEL)

Equality commutes with functorial interpretation of types.

- **Abstraction Theorem** (AT)

The interpretation $[\![t]\!]_1$ arises from $[\![t]\!]_0$.

Part III

Our work

joint work with

Patricia Johann, Neil Ghani,

Fredrik Nordvall-Forsberg and Tim Revell

Our work

Trying to capture parametricity axiomatically.

Use of bifibrational functorial semantics:

- It generalizes sets and relations over sets
- Enough to prove theorems axiomatically (initial algebras, ...)

Opens path to higher dimensional parametricity

Relations bifibrations

Definition (relations bifibration)

Given a bifibration $U : \mathcal{E} \to \mathcal{B}$

$\mathcal{B}$ *has products*

The **relations bifibration** *of* $\mathcal{E}$ *over* $\mathcal{B}$ *is* $Rel(U)$

$$\begin{array}{ccc} Rel(\mathcal{E}) & \xrightarrow{q} & \mathcal{E} \\ {\scriptstyle Rel(U)}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ \mathcal{B} \times \mathcal{B} & \xrightarrow[\_\times\_]{} & \mathcal{B} \end{array}$$

arising via change of base (pullback).

When U has right adjoint, we can use opfibrational properties to define the equality functor $\mathrm{Eq} \colon \mathcal{B} \to \mathsf{Rel}(\mathcal{E})$.

Generalized setting for relations

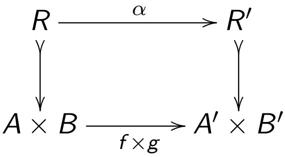
The category $\text{Rel}(\mathcal{E})$ has

- objects: triples (A, B, X) such that $U(X) = A \times B$
- morphisms: triples (f, g, h) such that $U(h) = f \times g$

We can think X as a relation over A and B.

`Rel` is the relations bifibration of $sub\colon \text{Sub}(\text{Set}) \to \text{Set}$:

- objects: $(A, B, R \subseteq A \times B)$
- morphisms: triples (f, g, α)



The framework

We want to define $\lambda 2$-fibrations $p \colon \mathcal{E} \to \mathcal{B}$.

The category $\mathcal{B}$ has natural numbers as objects.

The generic object is 1.

The product is given by the sum of natural numbers.

System F types as lifted functors

The objects in the fibre $\mathcal{E}_n$ are lifted functors

$$
\begin{array}{ccc}
|\mathrm{Rel}(\mathcal{E})|^n & \xrightarrow{\;[\![T]\!]_1\;} & \mathrm{Rel}(\mathcal{E}) \\
\Big\downarrow{\scriptstyle \mathrm{Rel}(U)} & & \Big\downarrow{\scriptstyle \mathrm{Rel}(U)} \\
|\mathcal{B}|^n \times |\mathcal{B}|^n & \xrightarrow[\;[\![T]\!]_0 \times [\![T]\!]_0\;]{} & \mathcal{B} \times \mathcal{B}
\end{array}
$$

Given a System F type judgement

$$\Gamma \vdash T \text{ type}$$

$[\![T]\!]$ is a lifted functor.

Identity Extension Lemma

Lemma (IEL, Reynolds-style)

If $\Gamma \vdash T$ type, then for every object $\bar{A}$ in Set^n

$$[\![T]\!]_1(\mathrm{Eq}^n\,\bar{A}) = \mathrm{Eq}\,([\![T]\!]_0\bar{A})$$

Lemma (IEL, bifibrationally)

If $\Gamma \vdash T$ type, then $[\![T]\!]$ is equality preserving, i.e., the following diagram commutes:

$$\begin{array}{ccc} |Rel(\mathcal{E})|^n & \xrightarrow{[\![T]\!]_1} & Rel(\mathcal{E}) \\ {\scriptstyle |\mathrm{Eq}|^n}\uparrow & & \uparrow{\scriptstyle \mathrm{Eq}} \\ |\mathcal{B}|^n & \xrightarrow[{[\![T]\!]_0}]{} & \mathcal{B} \end{array}$$

$\Rightarrow$ Equality preserving lifted functors

Terms are natural transformations

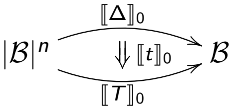
Term judgement

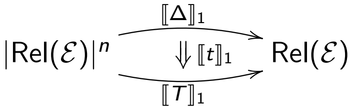
$$\Gamma; \Delta \vdash t \colon T$$

For every object $\bar{A}$ of $|\mathcal{B}|^n$ and $\bar{R}$ in $|\mathrm{Rel}(\mathcal{E})|^n$ family of morphisms

$$[\![t]\!]_0 \bar{A} \colon [\![\Delta]\!]_0 \bar{A} \to [\![T]\!]_0 \bar{A} \qquad [\![t]\!]_1 \bar{R} \colon [\![\Delta]\!]_1 \bar{R} \to [\![T]\!]_1 \bar{R}$$

$[\![\Delta]\!]$, $[\![T]\!]$ functors with discrete domain $\Rightarrow$ natural transformations





Abstraction Theorem

Consider a judgement $\Gamma; \Delta \vdash t : T$.

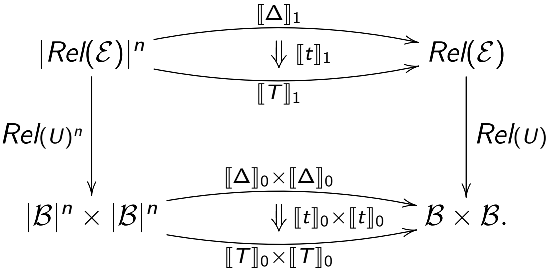
Theorem (Abstraction Theorem, Reynolds-Style)

Let $\bar{A}, \bar{B} \in Set^n$, $\bar{R} \in Rel^n(\bar{A}, \bar{B})$, *if* $(a, b) \in [\![\Delta]\!]_1 \bar{R}$ *then*

$$([\![t]\!]_0\ \bar{A}\ a, [\![t]\!]_0\ \bar{B}\ b) \ \in\ [\![T]\!]_1 \bar{R}$$

Theorem (Abstraction Theorem, bifibrationally)

$[\![t]\!] : [\![\Delta]\!] \to [\![T]\!]$ *defines a lifted natural transformation*



Part IV

Universal parametricity

joint work with

Neil Ghani and Fredrik Nordvall-Forsberg

Forall types in the base category

Polymorphic functions $\Rightarrow$ types application

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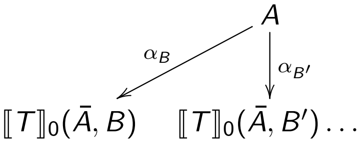
Polymorphic functions $\Rightarrow$ types application

$$\alpha_B f = f\ B$$

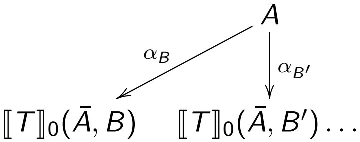


Forall types in the base category

Polymorphic functions $\Rightarrow$ types application

$$\alpha_B f = f\ B$$

Parametric condition $\Rightarrow$ application for $(f, f) \in \mathrm{Eq}(A)$



Forall types in the base category

Polymorphic functions $\Rightarrow$ types application

$$\alpha_B f = f\ B$$

Parametric condition $\Rightarrow$ application for $(f, f) \in \mathrm{Eq}(A)$

$$\eta_{R_{(A,B)}}(f, f) = (f\ A, f\ B)$$

$$
\begin{array}{ccc}
 & A & \\
\swarrow^{\alpha_B} & \downarrow^{\alpha_{B'}} & \\
[\![T]\!]_0(\bar{A}, B) & [\![T]\!]_0(\bar{A}, B') \ldots &
\end{array}
\qquad
\begin{array}{ccc}
 & \mathrm{Eq}(A) & \\
\swarrow^{\eta_R} & \downarrow^{\eta_{R'}} & \\
[\![T]\!]_1(\mathrm{Eq}(\bar{A}), R) & [\![T]\!]_1(\mathrm{Eq}(\bar{A}), R') \ldots &
\end{array}
$$

Forall types in the base category

Polymorphic functions $\Rightarrow$ types application

$$\alpha_B f = f\ B$$

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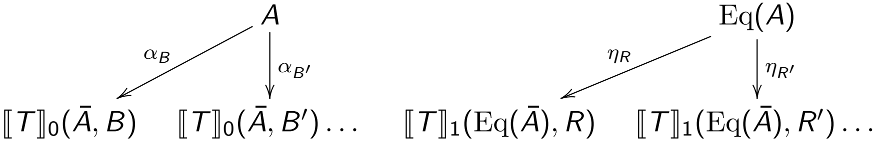
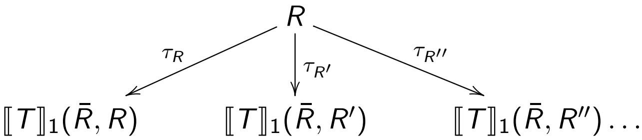


Lifted structure: η over (α, α).

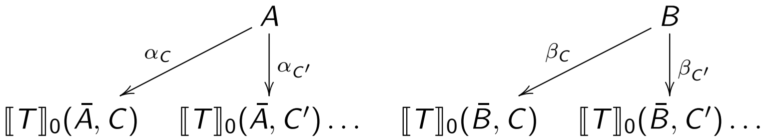
If it exists, the terminal cone is denoted $(\forall_0 \bar{A}, \alpha_{\bar{A}}, \eta_{\bar{A}})$.

Forall types in the total category

Types application τ



should "live over" the cones like before



The terminal cone is denoted

$$((\forall_0 \bar{A}, \alpha_{\bar{A}}, \eta_{\bar{A}}), (\forall_0 \bar{B}, \beta_{\bar{B}}, \xi_{\bar{B}}), \forall_1 \bar{R}, \tau_{\bar{R}}).$$

The functor $[\![\forall X.T]\!]$

Definition

If the terminal cones $(\forall_0 \bar{A}, \alpha_{\bar{A}}, \eta_{\bar{A}})$ and

$$((\forall_0 \bar{A}, \alpha_{\bar{A}}, \eta_{\bar{A}}), (\forall_0 \bar{B}, \beta_{\bar{B}}, \xi_{\bar{B}}), \forall_1 \bar{R}, \tau_{\bar{R}})$$

exist, we define

$$[\![\forall X.T]\!]_0 \bar{A} := \forall_0 \bar{A}$$

and

$$[\![\forall X.T]\!]_1 \bar{R} := \forall_1 \bar{R}.$$

Proposition

The functor $[\![\forall X.T]\!]$ gives an interpretation for the type judgment $\Gamma \vdash \forall X.T$ in accord with the definition via adjunction.

Related work

- Birkedal, Møgelberg *“Categorical models of parametric polymorphism”*
- Dunphy and Reddy *“Parametric limits”*
- Hermida, Reddy and Robinson *“Logical relations and parametricity - A Reynolds programme for category theory and programming languages”*
- Some unpublished work of Hermida
- Robinson, Rosolini *“Reflexive graphs and parametric polymorphism ”*

Thank you!