Homotopy-theoretic model of functional extensionality in the effective topos

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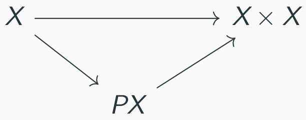
Homotopy type theory & model categories on topoi

Homotopy type theory

- Marin-Löf type theory supports an internal notion of equality given by the identity type $a, b : A \vdash \mathrm{Id}_A(a, b)$ type;
- Awodey and Warren: we can use model categories to interpret identity types:
  - Types $\Gamma \vdash X$ are interpreted as fibrations $X \to \Gamma$.
  - The identity type of X is obtained by factoring the map $X \xrightarrow{\Delta_X} X \times X$ as an acyclic cofibration, followed by a fibration.



Model structures on topoi

- There is a general way of endowing Grothendieck topoi with a model category structure (Cisinski 2002). In those "Cisinski model structures" the cofibrations are monomorphisms.
- This general method employs the small object argument.
- A big class of non-Grothendieck topoi are *realizability toposes*; no non-trivial model structure on those are known (to the author).
- In this talk: a model category structure on the subcategory of $\mathcal{E}ff$ giving rise to identity types w/ functional extensionality.

Assemblies

Definition

An *assembly* is a pair $(X, E_X) \in Asm$, where $E_X : X \to \mathcal{P}(\mathbb{N})$, s.t. $E_X(x)$ is always inhabited. A map $f : X \to Y$ is said to be a morphism of assemblies $(X, E_X) \to (Y, E_Y)$ if there is a computable ϕ such that $\phi(n) \in E_Y(f(x))$ whenever $n \in E_X(x)$.

- There is an embedding $\nabla : Set \to Asm$ that sends X to (X, E) with $E(x) = \mathbb{N}$.
- A *modest set* is an assembly (X, E) such that $E(x) \cap E(y) = \emptyset$ whenever $x \neq y$. There is an inclusion $Mod \hookrightarrow Asm$.

Effective topos

$\mathcal{E}\mathit{ff} := \mathit{Asm}_{ex/reg}$, i.e.

- Objects of the effective topos $\mathcal{E}\mathit{ff}$ are pairs $(X, \sim)$ where X is an assembly and $\sim$ is an (internal) equivalence relation $\sim \hookrightarrow X \times X$.
- A morphism $F\colon (X, \sim) \to (Y, \approx)$ is an (internal) functional relation between X and Y that respects $\sim$ and $\approx$.
- Then Asm is a full subcategory of $\mathcal{E}\mathit{ff}$: we identify an assembly X with $(X, \Delta_X : X \to X \times X)$.

Proposition (Hyland)

$\mathcal{E}\mathit{ff}$ is a topos with the natural numbers object $\mathbf{N}$ being an assembly $(\mathbb{N}, E)$ with $E(x) = \{x\}$.

The embedding $\nabla : \mathit{Set} \to \mathit{Asm}$ composes to $\nabla : \mathit{Set} \hookrightarrow \mathit{Asm} \hookrightarrow \mathcal{E}\mathit{ff}$.

Interval and homotopies in $\mathcal{E}ff$

Earlier work by Jaap van Oosten on the notion of homotopy in $\mathcal{E}ff$ establishes the *path object category* structure (in the sense of Van den Berg & Garner) on $\mathcal{E}ff$:

Theorem (Van Oosten)

There exists a pullback and product preserving functor $\mathsf{P} : \mathcal{E}ff \to \mathcal{E}ff$ *such that*

- *Any object X of $\mathcal{E}ff$ becomes an internal groupoid with* $\mathsf{P}X$ *being the object of arrows.*
- *There is a natural transformation* $\eta : \mathsf{P} \to \mathsf{PP}$ *sending a path to a contraction onto its endpoint*[1].

[1] Vacuum cleaner cord principle

Model structure from an interval object

(A framework inspired by Gambino & Sattler (arXiv:1510.00669v2)): Let $\mathcal{C}$ be a topos, and let I be an interval object with endpoints $\delta_0, \delta_1 : 1 \to I$ and connections $c_0, c_1 : I \times I \to I$.

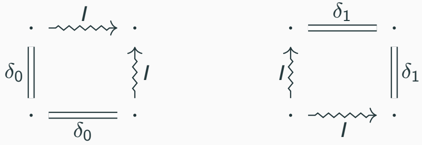


Figure 1: Visualisation of c_0 and c_1

- $c_0 \circ (\delta_1 \times I) = c_0 \circ (I \times \delta_1) = \mathrm{id}_I$
- $c_0 \circ (\delta_0 \times I) = c_0 \circ (I \times \delta_0) = \delta_0 \circ \epsilon$ where $\epsilon : I \to 1$
- $c_1 \circ (\delta_0 \times I) = c_1 \circ (I \times \delta_0) = \mathrm{id}_I$
- $c_1 \circ (\delta_1 \times I) = c_1 \circ (I \times \delta_1) = \delta_1 \circ \epsilon$

Fibrations

- For two maps $u : A \to B$, $i : C \to D$ we write $u \hat{\otimes} i$ for the canonical "inclusion" $(A \times D) \cup_{A \times C} (B \times C) \to B \times D$.
- A map is said to be a *fibration* if it has the RLP against maps of the form
  - $u \hat{\otimes} \delta_0 : (A \times I) \cup (B \times \{0\}) \to B \times I$, for $u : A \hookrightarrow B$
  - $u \hat{\otimes} \delta_1 : (A \times I) \cup (B \times \{1\}) \to B \times I$, for $u : A \hookrightarrow B$
- An object X is fibrant if $X \to 1$ is a fibration.
- Let $\mathcal{C}_f$ be a full subcategory of $\mathcal{C}$ on fibrant objects.

Homotopy equivalences

Definition

A *homotopy* $H : f \sim g$ is a map $H : I \times X \to Y$ such that $H \circ (\delta_0 \times X) = f$ and $H \circ (\delta_1 \times X) = g$.

A map f is a *homotopy equivalence* if there is a map g and homotopies θ, ϕ such that $\theta : \mathrm{id} \sim fg$ and $\phi : \mathrm{id} \sim gf$

- The homotopy relation is “well-behaved” on fibrant objects;
- Allows us to do basic abstract homotopy theory.

Model category structure

Theorem

If $\mathcal{C}$ is an elementary topos with an interval object I, then $\mathcal{C}_f$ carries a model category structure in which the cofibrations are monomorphisms, the fibrations are as above, and weak equivalences are homotopy equivalences. Furthermore, in this model category structure, $X \xrightarrow{X^{\epsilon}} X^I \to X \times X$ is a path object factorisation.

We do not require $\mathcal{C}$ or $\mathcal{C}_f$ to be cocomplete; we can directly apply this theorem to the effective topos.

Model category structure on $\mathcal{E}ff_f$

Applying the general construction to the effective topos with $I := \nabla(2)$ we get

Theorem

The category $\mathcal{E}ff_f$ possesses a model category structure in which the cofibrations are monomorphisms, the fibrations are as above, and weak equivalences are homotopy equivalences. In this model structure, $X \to X^I \to X \times X$ is a path object factorisation of X.

- This induces a model of TT with Id, Π, Σ, $+$, $\times$, 0, 1;
- The model supports functional extensionality;
- Not every object can be interpreted as a type – we have to restrict to fibrant objects.

Functional extensionality

Functional extensionality for simple exponents

Follows from basic categorical reasoning:

$[X, Y^I] \simeq [X \times I, Y] \simeq [I, Y^X]$

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Functional extensionality for dependent products

- (Proposition 4.9.5 from the HoTT book), functional extensionality for dependent products is equivalent to: let p be a fibration and f be a trivial fibration, then $\Pi_p(f)$ is a trivial fibration.
- Also equivalent to: cofibrations are stable under pullbacks along fibrations.

Thus, the induced model of TT supports functional extensionality.

Properties of the model

Discrete objects and discrete spaces i

Definition

An object $(X, \sim)$ of the effective topos is said to be *discrete* if it is a quotient of a subobject of $\mathbf{N}$.

N.B. Modest sets are discrete.

Discrete objects can be seen as "discrete spaces":

Proposition (Van Oosten)

An object X of $\mathcal{E}ff$ is discrete iff every path $I \to X$ is trivial.

As a consequence of this fact:

Proposition

Every discrete object of $\mathcal{E}ff$ is fibrant.

Discrete objects and discrete spaces ii

Since types are interpreted as fibrant objects, we already have a lot of examples of types in our model.

- The natural numbers object **N** in the effective topos is a modest set; by the previous proposition we have that the induced model of type theory supports natural numbers.
- The terminal object 1 is a modest set; thus it is a type.
- In fact all finite types are modest sets,

Because every path in a discrete object is trivial, there are no non-trivial homotopy equivalences between discrete objects. In particular that implies $Ho(Mod_f) = Ho(Mod) \simeq Mod$.

Contractible objects and trivial fibrations

Definition

A fibrant object X is said to be *contractible* if the fibration $X \to 1$ is acyclic. In category theoretic terms such object is called *injective*.

For instance, Ω and all power-objects are injective.

Definition

An object $(X, \sim)$ in the effective topos is said to be *uniform* if there is an epimorphism $\nabla Z \to (X, \sim)$.

Trivial fibrations and uniform objects

We have the following correspondence (assuming AoC & restricted LEM):

Uniform assembly	$\Longleftrightarrow$	Contractible assembly
Uniform epimorphism between assemblies	$\Longleftrightarrow$	Trivial fibration between assemblies
Uniform object	$\not\Longrightarrow$ $\Longleftarrow$	Contractible object
Fibrant uniform object	$\Longleftrightarrow$	Contractible object

(Uniform objects are quotients of $\neg\neg$-sheaves in $\mathcal{E}\mathit{ff}$; uniform maps are quotients of $\neg\neg$-sheaves in the slice topos.)

Restricting to assemblies i

- The characterisation of trivial fibrations between assemblies allows us to describe the homotopy category of assemblies.
- Given an assembly X: X is fibrant $\iff$ $X^I \xrightarrow{s} X$ is a trivial fibration $\iff$ s is a uniform epimorphism.
- Unfolding the definitions we arrive at: X is a fibrant assembly iff there is a computatble ϕ such that for each x and each $n \in E(x)$, $\phi(n)$ uniformly realizes all the elements in the "path-connected component" of x.
- Using this characterisation of assemblies one can also show that X^I is homotopic to Jaap's path object $\mathrm{P}X$, for a fibrant assembly X.

Restricting to assemblies ii

- The category of discrete objects is *reflective* in $\mathcal{E}ff$.

Proposition

Given a fibrant assembly X, the discrete reflection $\eta_X : X \to X_d$ is a homotopy equivalence. (Assuming AoC in the ambient set theory).

- This implies: $Ho(Asm_f) \simeq Ho(Mod) \simeq Mod$.

Conclusions

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Theorem

There is a model category structure on $\mathcal{E}ff_f$ in which the cofibrations are monomorphisms. The model category structure on $\mathcal{E}ff_f$ induces a model of dependent type theory with identity types, $+$, $\times$, Σ, Π, and all finite types. The identity type of a type X is given by X^I, and the resulting model supports functional extensionality.

Open problems

- Realizability/recursion-theoretic description of fibrant objects;
- The model structure on $\mathcal{E}ff$ as whole.
- HITs;
- Characterization of h-sets (in general, h-levels);
- Universal fibration.

Thank you for listening!