Higher Dimensional Parametricity

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- **Question:** Can we have proof-relevant parametricity?
  - Certainly sounds nice re intensional MLTT and HoTT
  - Should witnesses of such parametricity be parametric too?
- **Answer:** Yes and yes. And iterate!
  - Leads to HD parametricity using HD relations $\Rightarrow$ HD nats.
  - Parallels with HoTT as cubical methods arise naturally
- **Details:** Non-trivial technical and conceptual aspects:
  - Need strict CCC/$\forall$-structure.
  - Must preserve more than equalities, but not possible strictly

1D Parametricity

- **System F:** Define judgements $\Gamma \vdash T : *$ by
  - Variables: $X_1, ..., X_n \vdash X_i : *$
  - ArrowTypes: If $\Gamma \vdash T, U : *$ then $\Gamma \vdash T \to U : *$
  - Forall types: If $\Gamma, X \vdash T : *$ then $\Gamma \vdash \forall X.T : *$
  - Judgements for terms: $\Gamma; \Delta \vdash t : T$ where $\Gamma \vdash T : *$ and $(x_i : T_i) \in \Delta \Rightarrow \Gamma \vdash T_i : *$

- **John Reynolds:** Two inter-related semantics:

$$
\begin{array}{lcl}
[\![\Gamma \vdash T : *]\!]_0 & \in & \mathsf{Set}^{|\Gamma|} \to \mathsf{Set} \\
[\![\Gamma \vdash T : *]\!]_1 & \in & \forall A_1, A_2 \in \mathsf{Set}^{|\Gamma|}. \\
 & & \mathsf{Rel}^{|\Gamma|}(A_1, A_2) \to \mathsf{Rel}([\![\Gamma \vdash T : *]\!]_0 A_1, [\![\Gamma \vdash T : *]\!]_0 A_2)
\end{array}
$$

- **Arrow Types:** If $\Gamma \vdash T \to U : *$, use exponentials in Set and Rel

$$\begin{array}{rcl} [\![T \to U]\!]_0 A & = & [\![T]\!]_0 A \to [\![U]\!]_0 A \\ (f,g) \in [\![T \to U]\!]_1 R & \text{iff} & (a,b) \in [\![T]\!]_1 R \Rightarrow (fa, gb) \in [\![U]\!]_1 R \end{array}$$

Related functions map related inputs to related outputs.

- **Forall Types:** If $\Gamma \vdash \forall X.T : *$, then

$$\begin{array}{rcl} [\![\forall X.T]\!]_0 A & = & \{f : (S : \mathsf{Set}) \to [\![T]\!]_0 (A, S) \mid \\ & & \quad R : \mathsf{Rel}(A,B) \Rightarrow (fA, fB) \in [\![T]\!]_1 (EqA, R)\} \\ (f,g) \in [\![\forall X.T]\!]_1 R & \text{iff} & R' \in \mathsf{Rel}(A,B) \Rightarrow (fA, gB) \in [\![T]\!]_1 (R, R') \end{array}$$

- **Parametric Polymorphic Functions:** A uniformity constraint.

- **Identity Extension Lemma:** If $\Gamma \vdash T : *$ then $[\![T]\!]_1(\mathsf{Eq}S) = \mathsf{Eq}([\![T]\!]_0 S)$. Strict equality makes the inductive proof smooth.

Proof-Relevant Parametricity

Proof Relevance for Arrow types

- **Question:** What is a proof relevant relation
  - A function $R \to A \times B$ or $R : A \times B \to \mathsf{Set}$.

- **Arrow Types:** What is the logical relation? Handle-Turning suggests

$$\begin{aligned} [\![T \to U]\!]_0 A &= [\![T]\!]_0 A \to [\![U]\!]_0 A \\ [\![T \to U]\!]_1 R(f,g) &= (a : A)(b : B) \to [\![T]\!]_1 R(a,b) \to [\![U]\!]_1 R(fa, gb) \end{aligned}$$

- **Justification:** We can prove the IEL
  - Further justified as this is the closed structure of proof relevant relations.

- **Idea:** Handle-turn the proof-irrelevant interpretation of ∀-types

$$
\begin{aligned}
[\![\forall X.T]\!]_0 A \;&=\; \{(f_0, f_1) | \\
&\quad f_0 : (S : \mathsf{Set}) \to [\![T]\!]_0(A, S) \\
&\quad f_1 : (R : \mathsf{Rel}(A, B)) \to [\![T]\!]_1(EqA, R)(fA, fB)\}
\end{aligned}
$$

$$
[\![\forall X.T]\!]_1 R\,(f_0, f_1)(g_0, g_1) \;=\; (R' : \mathsf{Rel}(A, B)) \to [\![T]\!]_1(R, R')(fA, gB)
$$

A Proof Relevant interpretation of $\forall$-types

- **Idea:** Handle-turn the proof-irrelevant interpretation of $\forall$-types

$$\begin{aligned} [\![\forall X.T]\!]_0 A &= \{(f_0, f_1) | \\ & \quad f_0 : (S : \mathsf{Set}) \to [\![T]\!]_0(A, S) \\ & \quad f_1 : (R : \mathsf{Rel}(A, B)) \to [\![T]\!]_1(EqA, R)(fA, fB)\} \end{aligned}$$

$$[\![\forall X.T]\!]_1 R\,(f_0, f_1)(g_0, g_1) \;=\; (R' : \mathsf{Rel}(A, B)) \to [\![T]\!]_1(R, R')(fA, gB)$$

- **Problems:** Many problems ...
  - Can't prove IEL
  - The function f_1 is ad-hoc polymorphic
  - No constraints on f_1 and g_1 in $[\![\forall X.T]\!]_1 R\,(f_0, f_1)(g_0, g_1)$

Proof-Relevant Parametricity $\Rightarrow$ *2D Parametricity*

Use 2-Relations to Make the uniformity condition, uniform

- **Idea:** Add a layer of 2-relations to constrain the relational layer
  - And functions $[\![T]\!]_2 : |2\text{–}\mathsf{Rel}|^n \to 2\text{–}\mathsf{Rel}$ living over $[\![T]\!]_1$
  - And $\mathsf{Eq}_2 : \mathsf{Rel} \to 2\text{–}\mathsf{Rel}$, prove $\mathsf{Eq}_2(R \to R') = \mathsf{Eq}_2 R \to \mathsf{Eq}_2 R'$
  - Prove expanded IEL: $[\![T]\!]_2(\mathsf{Eq}_2 R) = \mathsf{Eq}_2([\![T]\!]_1 R)$

- **Question:** So, what is a 2-relation?
  - Ans 1: $2\text{–}\mathsf{Rel}(R, R')(R, R') = \mathsf{Rel}(\{R\}, \{R'\})$?
  - Can't prove IEL ...
  - No relationship between what related proofs relate.

2-Relations Defined

- **Defn:** Let $\mathsf{sp} = (l \leftarrow m \rightarrow r)$.

$$\mathsf{Rel} = \mathsf{Set}^{sp} \qquad 2\text{–}\mathsf{Rel} = \mathsf{Set}^{\mathsf{sp}^2}$$

with proof-irrelevant versions.

- **Detail:** Pictorially ....

$$\begin{array}{ccccc}
A_{lr} & \longleftarrow & A_{mr} & \longrightarrow & A_{rr} \\
\uparrow & & \uparrow & & \uparrow \\
A_{lm} & \longleftarrow & A_{mm} & \longrightarrow & A_{rm} \\
\downarrow & & \downarrow & & \downarrow \\
A_{ll} & \longleftarrow & A_{ml} & \longrightarrow & A_{rl}
\end{array}$$

$$\begin{aligned} A_{mm} \ \subseteq \ & \boldsymbol{\Sigma}(a_{ll} : A_{ll})(a_{lr} : A_{lr})(a_{rl} : A_{rl})(a_{rr} : A_{rr}). \\ & \quad A_{lm}(a_{ll}, a_{lr}) \times A_{ml}(a_{ll}, a_{rl}) \times A_{mr}(a_{lr}, a_{rr}) \times A_{rm}(a_{rl}, a_{rr}) \end{aligned}$$

- **Intuition:** A_{mm} chooses a subset of the boundary elements

- **Equalities:** 2-relations have four projections and two equalities!

$$\begin{aligned}\mathsf{Eq}_{||}, \mathsf{Eq}_{=} &: \mathsf{Rel} \to 2\text{–}\mathsf{Rel} \\ \mathsf{Eq}_{||} R &= \backslash(i,j) \to Ri \\ \mathsf{Eq}_{=} R &= \backslash(i,j) \to Rj\end{aligned}$$

- **Pictorially:** The image of $R : \mathsf{Rel}(A, B)$ under $\mathsf{Eq}_{=}, \mathsf{Eq}_{||}$ is

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$$
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$$
\begin{array}{ccccc}
A & \leftarrow & R & \rightarrow & B \\
\uparrow & & \uparrow & & \uparrow \\
A & \leftarrow & R & \rightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
A & \leftarrow & R & \rightarrow & B
\end{array}
\qquad\qquad
\begin{array}{ccccc}
B & \leftarrow & B & \rightarrow & B \\
\uparrow & & \uparrow & & \uparrow \\
R & \leftarrow & R & \rightarrow & R \\
\downarrow & & \downarrow & & \downarrow \\
A & \leftarrow & A & \rightarrow & A
\end{array}
$$

- **Next Guess:** Make parametricity parametric

$$
\begin{aligned}
[\![\forall X.T]\!]_0 A \;=\; & \{(f_0, f_1) | \\
& f_0 : (S : \mathsf{Set}) \to [\![T]\!]_0(A, S) \\
& f_1 : (R : \mathsf{Rel}(A, B)) \to [\![T]\!]_1(\mathsf{Eq}A, R)(fA, fB) \\
& \forall Q : 2\text{–}\mathsf{Rel}.(f_1 Q_{lm}, f_1 Q_{ml}, f_1 Q_{mr}, f_1 Q_{rm}) \in \\
& \qquad [\![T]\!]_2(\mathsf{Eq}A, Q)(f_0 Q_{ll}, f_0 Q_{lr}, f_0 Q_{rl}, f_0 Q_{rr})\}
\end{aligned}
$$

- **Note:** Thankfully, only one functor Set $\to$ 2–Rel. Means only one 2-d clause in the above logical relation

- **Key Idea:** Identity extension cant yet be proven ... need connections C : Rel $\to$ 2–Rel

$$\mathsf{C}\; R\; (i, j) = R(i \vee j) \quad (i, j \in \{l \leq m \leq r\})$$

- **Level 1:** Can now define $[\![\forall X.T]\!]_1 \; R \; ((f_0, f_1),)(g_0, g_1))$

$$
\begin{aligned}
&\{\phi : (R' : \mathsf{Rel}) \to [\![T]\!]_1(R, R')(f_0 R'_l, g_0 R'_r) | \\
&\forall Q : 2\text{–}\mathsf{Rel}.(f_1 Q_{ml}, \phi Q_{lm}, g_1 Q_{mr}, \phi Q_{rm}) \in \\
&\qquad\qquad [\![T]\!]_2(\mathsf{Eq}_{||} R, Q)(f_0 Q_{ll}, f_0 Q_{rl}, g_0 Q_{lr}, g_0 Q_{rr}) \\
&\wedge \\
&\forall Q : 2\text{–}\mathsf{Rel}.(\phi Q_{ml}, f_1 Q_{lm}, \phi Q_{mr}, g_1 Q_{rm}) \in \\
&\qquad\qquad [\![T]\!]_2(\mathsf{Eq}_{||} R, Q)(f_0 Q_{ll}, f_0 Q_{rl}, g_0 Q_{lr}, g_0 Q_{rr}) \\
&\wedge \\
&\forall Q : 2\text{–}\mathsf{Rel}.(f_1 Q_{ml}, f_1 Q_{lm}, \phi Q_{mr}, \phi Q_{rm}) \in \\
&\qquad\qquad [\![T]\!]_2(\mathsf{C} R, Q)(f_0 Q_{ll}, f_0 Q_{rl}, f_0 Q_{lr}, g_0 Q_{rr}) \\
&\}
\end{aligned}
$$

- **Level 2:** Straightforward: $(\phi_0, \phi_1, \phi_2, \phi_3) \in ([\![\forall X.T]\!]_2 Q)(f, g, l, h)$iff

$$
\begin{aligned}
&(\forall Q' : 2\text{–}\mathsf{Rel})(\phi_0 Q'_{ml}, \phi_1 Q'_{lm}, \phi_2 Q'_{mr}, \phi_3 Q'_{rm}) \in \\
&\qquad\qquad [\![T]\!]_2(Q, Q')(f_0 Q'_{ll}, g_0 Q'_{rl}, h_0 Q'_{lr}, l_0 Q'_{rr})
\end{aligned}
$$

HD Logical Relations .... More Abstraction Igor!

Difficulties facing a HD-generalisation

- **Notation:** Combinatorial explosion in number/length of clauses
  - Use relations as spans as opposed to indexed sets of proofs
  - Develop the cubical-algebra of iterated spans
- **Fibred Structures:** Various structure needs to be over various other structures
  - Higher dimensional actions must be STRICTLY over lower dimensional actions.
  - Witnesses of higher dimensional relatedness need to be over witnesses of lower dimensional relatedness
  - IEL for higher dimensions need to be over IEL for lower dimensions: logical relation to act on isomorphisms

- **Defn:** n–Rel $=$ Set$^{\mathsf{sp}^n}$. Face maps, degeneracies, connections and transpositions

$$\begin{array}{rcl}
\delta_i^d & : & \mathsf{sp}^n \to \mathsf{sp}^{n+1} \quad (d \in l, r) \\
\delta_i^d\,(x_1, \cdots, x_n) & = & (x_1, \cdots, x_i, d, x_{i+1,\ldots} x_n) \\
\epsilon_i & : & \mathsf{sp}^n \to \mathsf{sp}^{n-1} \\
\epsilon_i\,(x_1, \cdots, x_n) & = & (x_1, \cdots, x_i, x_{i+2,\ldots} x_n) \\
\gamma_i & : & \mathsf{sp}^n \to \mathsf{sp}^{n-1} \\
\gamma_i\,(x_1, \cdots, x_n) & = & (x_1, \cdots, x_i, x_{i+1} \vee x_{i+2}, x_{i+3}, \cdots, x_n) \\
\tau_i & : & \mathsf{sp}^n \to \mathsf{sp}^n \\
\tau_i\,(x_1, \cdots, x_n) & = & (x_1, \cdots, x_i, x_{i+2}, x_{i+1}, x_{i+3}, \cdots, x_n)
\end{array}$$

- **Defn:** $\Box$ is the category of natural numbers and such maps, $\Box_\delta$ has only face maps. Precomposition defines

$$\Box^{op} \to \mathsf{Cat} \quad \text{where} \quad n \mapsto n\text{–Rel}, \;\; \mu : m \to n \;\mapsto\; \_ \circ \mu : n\text{–Rel} \to m\text{–Rel}$$

- **Localisation:** We need to generalise to higher dimensions STRICT requirements such as

$$[\![T]\!]_1(R : \mathsf{Rel}(A, B)) : \mathsf{Rel}([\![T]\!]_0 A, [\![T]\!]_0 B)$$

- **Idea:** Define for $x \in \mathsf{sp}^n$, $|x| = \#\{i | x_i = m\}$. Then

$$\delta_x : \mathsf{sp}^{|x|} \to \mathsf{sp}^n$$

embeds a low dimensional cube into a higher dimensional one. Thus, extracts from an n-relation and node within it, the relation centred at that node.

- **Formalisation:** We require strict equality

$$[\![T]\!]_n A \circ \delta_x = [\![T]\!]_{|x|}(A \circ \delta_x)$$

- **CCC-structure is easy?** Since $n-\mathsf{Rel}$ is a presheaf category, and arrow types are interpreted as exponential, we just use the presheaf exponential?

$$(G^F)X = \mathsf{Nat}(\mathcal{C}(X, -) \times F, G) : \mathcal{C} \to \mathsf{Set}$$

  Wrong! This doesn't give the strictness required

- **Solution:** Given presheaves F, G over $\mathcal{C}$

$$(G^F)X = \mathsf{Nat}(F \circ \mathsf{cod}_x, G \circ \mathsf{cod}_x) : X/\mathcal{C} \to \mathsf{Set}$$

- **Concretely:** Codomain functor IS δ_x
  - Unwinding means we get exponentials on all the projections
  - and a function between each cube's core respecting those projections

- **First Guess:** 0 is the initial object in $\square$, so sth like

$$
\begin{array}{rcl}
[\![\forall X.T]\!]_0 A & = & \{(f_0, f_1, f_2, \cdots) | \\
& & f_0 : (S : \mathsf{Set}) \to [\![T]\!]_0(A, S) \\
& & f_1 : (R : \mathsf{Rel}) \to [\![T]\!]_1(\epsilon_1 A, R)(\vec{f_0}\vec{\delta}R) \\
& & f_2 : (Q : 2\text{–}\mathsf{Rel}) \to [\![T]\!]_2(\epsilon_2 A, Q)(\vec{f_1}\vec{\delta}Q) \\
& & f_3 : (P : 3\text{–}\mathsf{Rel}) \to [\![T]\!]_3(\epsilon_3 A, P)(\vec{f_2}\vec{\delta}P) \\
& & ... \\
& & \}
\end{array}
$$

f_{i+1} witnesses the parametricity/uniformity of f_i

- **Comments:** Not bad, but LHS is a functor and no sign of how to use connections or transpositions

  – And we simply can't have dots in our theory! Must internalise

- **Key Idea:** Cubical algebra controlled complexity of each clause
  - Now, control the number of clauses by internalising them via the cube category.

- **Defn:** The higher dimensional logical relation for $\forall$-types is

$$\begin{aligned}[\![\forall X.T]\!]_n\ A\ x \ = \ \{\phi : (\Pi k \geq 0)(\Pi \mu : \Box(k, |x|))(\Pi Q : k-\mathsf{Rel}) \\ \mathsf{Nat}(1, [\![T]\!]_k(\vec{\mu}\vec{\delta_x}A, Q)) \quad | \\ \forall(k, \mu, Q, \mu').\ \phi\ k\ f\ \mu \cdot \mu' = \phi(\mathsf{dom}\mu')(\mu\mu')(\vec{\mu'}Q)\}\end{aligned}$$

- **Intuition:** Logical relation defined simultaneously at all levels
  - The first clause gives parametricity witnesses at all levels and maps in $\Box$
  - The second clause ensures we are over the right witnesses

What's done, what's not done

- **Done:** An interpretation of types via a HD logical relation
  - Each layer is strictly over previous layers
  - IEL proved upto isos over lower dimensional isos
  - Tamed combinatorial complexity, identified significant uniformity
- **Not Done:** A full model is a split-$\lambda 2$-fibration
  - Have all parts but Beck-Chevellay holds only upto iso.
  - That is, types equal under substitutions are isomorphic. Bad!
  - Looking at strictification, explicit substitutions etc.

Conclusions

- **Theory:** Much remains to do ....
  - Finish off the model, get substitution right
  - Use a HoTT meta-theory
  - Extend to Dependent Types

- **Applications:** Investigate applications and utility
  - Higher naturality and higher dinaturality
  - More generally, theorems and coherence for free
  - Internal parametricity