Normalization and the Taylor expansion of non-uniform λ-terms

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Disclaimer

I will lie to you.

A version without (too many) lies is avaible:
`http://iml.univ-mrs.fr/~vaux/pub/taylornf.pdf`.

Quantitative semantics

A prime aged idea (Girard, '80s, before LL)

λ-terms = analytic functions = power series
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Reformulate q.s. in a typed setting (LL) using standard algebra:

- types $\rightsquigarrow$ particular topological vector spaces (or semimodules):
  $[\![A]\!] \subseteq \mathbb{S}^{|A|}$ + some additional structure
- function terms $\rightsquigarrow$ power series

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Differentiation of λ-terms (Ehrhard-Regnier 2003-2004)

So we can *differentiate* λ-terms, and compute their Taylor expansion!
And one can mimick that in the syntax:

- differential λ-calculus
- a finitary fragment: resource λ-calculus
  = the target of Taylor expansion

Resource λ-calculus

Resource terms

$$\begin{array}{rcccl} \Delta & \ni & s, t, \ldots & ::= & x \mid \lambda x\, s \mid \langle s \rangle\ \overline{t} \\ !\Delta & \ni & \overline{s}, \overline{t}, \ldots & ::= & [s_1, \ldots, s_n] \end{array}$$

Meaning: $\langle s \rangle\ [s_1, \ldots, s_n] = (Ds)_0 \cdot (s_1, \ldots, s_n)$

Resource reduction

$$\langle \lambda x\, s \rangle\ \overline{t} \to_\partial \partial_x s \cdot \overline{t} \quad \text{(anywhere)}$$

$$\partial_x s \cdot \overline{t} = \begin{cases} \sum_{f \in \mathfrak{S}_n} s\left[t_{f(1)}, \ldots, t_{f(n)}/x_1, \ldots, x_n\right] & \text{if } \mathbf{n}_x(s) = \#\overline{t} = n \\ 0 & \text{otherwise} \end{cases}$$

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linearity: $\lambda x\, 0 = 0$, $\langle s \rangle\ [t_1 + t_2, u] = \langle s \rangle\ [t_1, u] + \langle s \rangle\ [t_2, u]$, …

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- Resource reduction preserves free variables, is size-decreasing, strongly confluent and normalizing.

Taylor expansion of λ-terms

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$$\mathcal{T}((M)\ N) = \sum_{n \in \mathbf{N}} \frac{1}{n!} \langle \mathcal{T}(M) \rangle\ \mathcal{T}(N)^n$$

$$\mathcal{T}(x) = x \qquad \mathcal{T}(\lambda x\, M) = \lambda x\, \mathcal{T}(M)$$

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Theorem (Ehrhard-Regnier, CiE 2006)

If $M \in \Lambda$, then $\mathcal{T}(M)$ normalizes to $\mathcal{T}(\mathcal{BT}(M))$.

Moral

In the ordinary λ-calculus $\mathcal{BT}(M) \simeq \mathsf{NF}(\mathcal{T}(M))$.

Normalizing Taylor expansions

But how can $\mathcal{T}(M)$ even normalize?

We want to set

$$\mathsf{NF}(\mathcal{T}(M)) = \sum_{s \in \Delta} \mathcal{T}(M)_s . \mathsf{NF}(s)$$

$\rightsquigarrow$ infinite sums (and in general we might consider all kinds of coefficients)
$\rightsquigarrow$ convergence?

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Theorem (Ehrhard-Regnier 2004, published in TCS in 2008)

Write $|\mathcal{T}(M)| = |\mathcal{T}(M)|$. Then for all $t \in \Delta$, there is at most one $s \in |\mathcal{T}(M)|$ such that $\mathsf{NF}(s)_t \neq 0$.

Proof.

λ-terms are uniform: their finitary approximants are pairwise coherent. □

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This fails in general

$\mathsf{NF}\left(\sum_{n \in \mathbf{N}} \langle \lambda x\, x \rangle^n [y]\right) = ?$ $\qquad$ $\langle \lambda x\, x \rangle^n [y] = \langle \lambda x\, x \rangle \, [\langle \lambda x\, x \rangle \, [\cdots [y] \cdots]]$

A generic quantitative non-uniform calculus

$$\mathbb{S}\left[\Lambda_{\mathbb{S}}\right] \ni M, N, \ldots ::= x \mid \lambda x\, M \mid (M)\ N$$

$$(\lambda x\, M)\ N \to_{\beta} M\left[N/x\right]$$

A generic quantitative non-uniform calculus

$$\mathbb{S}\,[\Lambda_{\mathbb{S}}] \ni M, N, \ldots ::= x \mid \lambda x\, M \mid (M)\ N \mid M + N$$

$$(\lambda x\, M)\ N \to_\beta M\,[N/x] \qquad\qquad (M+N)\ P = (M)\ P + (N)\ P$$

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Taylor expansion in a non uniform setting

$$\mathcal{T}\left(\sum_{i=1}^{n} a_i\, M_i\right) = \sum_{i=1}^{n} a_i\, \mathcal{T}\,(M_i)$$

A ~~generic~~ minimalistic quantitative non-uniform calculus

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Example

Let $\delta_M = \lambda x\; (M + (x)\; x)$ and $\infty_M = (\delta_M)\; \delta_M$: $\infty_M \to_\beta{}^* M + \infty_M$.

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Example

Let $\delta_M = \lambda x\ (M + (x)\ x)$ and $\infty_M = (\delta_M)\ \delta_M$: $\infty_M \rightarrow_\beta{}^* M + \infty_M$.
Then $\mathsf{NF}\left(\mathcal{T}\left(\infty_M\right)\right) = ?$

Finiteness structures to the rescue!

The main artifact of finiteness spaces:

Definition

- If $a, a' \subseteq A$, write $a \perp a'$ iff $a \cap a'$ is finite.
- If $\mathfrak{S} \subseteq \mathfrak{P}(A)$, let $\mathfrak{S}^\perp := \{a' \subseteq A \; ; \forall a \in \mathfrak{S},\; a \perp a'\}$.
- A finiteness structure is any $\mathfrak{F} = \mathfrak{S}^\perp$.

When is $\mathcal{T}(M)$ normalizable?

- Write $s \geq_\partial t$ if $s \to_\partial{}^* t + \cdots$.
- Let $\uparrow t = \{s \in \Delta \; ; s \geq_\partial t\}$.
- $\mathcal{T}(M)$ is normalizable iff for all normal $t \in \Delta$, $|\mathcal{T}(M)| \perp \uparrow t$.
- $\{\uparrow t \; ; t \text{ normal } \in \Delta\}^\perp$ is the finiteness structure of (supports of) normalizable vectors.

Typed terms have a finitary Taylor expansion

Let system F_+ be system F plus $\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash M + N : A}$.

Theorem (Ehrhard, LICS 2010)

If $M \in \mathbb{S}[\Lambda_{\mathbb{S}}]$ is typable in system F_+, then $|\mathcal{T}(M)| \in \{\uparrow t \,;\, t \in \Delta\}^{\perp}$.

Proof.

Manage sets of resource terms as if they were λ-terms, and follow the usual reducibility technique, associating a finiteness structure $\mathfrak{F}(A) \subseteq \{\uparrow t \,;\, t \in \Delta\}^{\perp}$ with each type A.

□

Normalizability = Finiteness

Pagani-Tasson-V., FoSSaCS 2016

- Typability in F can be relaxed to strong normalizability.
- Then the implication

$$M \in \mathsf{SN} \Rightarrow |\mathcal{T}(M)| \in \{\uparrow t \,;\, t \in \Delta\}^{\perp}$$

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- and head normalizability.

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This equivalence is nice...

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Normalizable terms have a normalizable Taylor expansion.

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Theorem?

If $M \to_\beta N$ then $\mathcal{T}(M) \rightsquigarrow \mathcal{T}(N)$.

β-reduction through Taylor expansion

Recall that:

$$\mathcal{T}\left((M)\ N\right) = \sum_{n \in \mathbf{N}} \frac{1}{n!} \left\langle \mathcal{T}\left(M\right) \right\rangle \mathcal{T}\left(N\right)^{n}$$

In quantitative semantics:

$$\llbracket (\lambda x\, M)\ N \rrbracket = \llbracket M\,[N/x] \rrbracket = \sum_{n \in \mathbf{N}} \frac{1}{n\,!} \left(\frac{\partial^{n} \llbracket M \rrbracket}{\partial x^{n}} \right)_{x=0} \cdot \llbracket N \rrbracket^{n}.$$

β-reduction through Taylor expansion: key steps

Promotion

$$\sigma^! := \sum_{n \in \mathbf{N}} \frac{1}{n\,!} \sigma^n \in \mathbb{S}^{!\Delta}$$

for all $\sigma \in \mathbb{S}^\Delta$

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Reduction

$$\langle \lambda x\, \sigma \rangle\ \tau^! \rightsquigarrow \partial_x \sigma \cdot \tau^!$$

and more generally

$$\langle \lambda x\, \sigma \rangle\ \overline{\tau} \rightsquigarrow \partial_x \sigma \cdot \overline{\tau} = \sum_{s \in \Delta, \overline{t} \in !\Delta} \sigma_s \overline{\tau}_{\overline{t}}\, \partial_x s \cdot \overline{t}$$

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Substitution

$$\mathcal{T}\left(M\left[N/x\right]\right) = \partial_x \mathcal{T}\left(M\right) \cdot \mathcal{T}\left(N\right)^!$$

β-reduction through Taylor expansion, step 1: problem

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Is this sum always defined?

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Is this sum always defined?
Yes (easy and already known): given $\bar{t} \in !\Delta$ and $n = \#\bar{t}$, there are finitely many $(t_1, \ldots, t_n)$ such that $\bar{t} = [t_1, \ldots, t_n]$.

β-reduction through Taylor expansion, step 2: problem

Reduction

$$\langle \lambda x\, \sigma \rangle\ \overline{\tau} \rightsquigarrow \partial_x \sigma \cdot \overline{\tau} = \sum_{s \in \Delta, \overline{t} \in !\Delta} \sigma_s \overline{\tau}_{\overline{t}}\, \partial_x s \cdot \overline{t}$$

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Yes (used by Ehrhard):

Lemma

If $s \succ_\partial t$ *(i.e.* $s \rightarrow_\partial t + \ldots$*) then* $\mathbf{s}\,(s) \leq 2\mathbf{s}\,(t) + 2$.

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Is this notion of reduction sufficient?
No: we need to reduce arbitrarily many redexes in parallel!

$$\mathcal{T}\,((y)\ (\lambda x\, x)\ z) \quad = \quad \sum_{n, k_1, \ldots, k_n \in \mathbf{N}} \frac{1}{n! k_1! \cdots k_n!} \langle y \rangle\ \left[\langle \lambda x\, x \rangle\ z^{k_1}, \ldots, \langle \lambda x\, x \rangle\ z^{k_n} \right]$$

$$\mathcal{T}\,((y)\ z) \quad = \quad \sum_{n \in \mathbf{N}} \frac{1}{n!} \langle y \rangle\ z^n$$

Parallel reduction on vectors of resource terms

Write $s \Rightarrow_\partial \sigma' \in \mathbf{N}\left[(!)\Delta\right]$ for the parallel reduction of resource terms.

Definition

Write $\sigma \mathrel{\widetilde{\Rightarrow_\partial}} \sigma'$ if:

$$\sigma = \textstyle\sum_{i \in I} a_i\, s_i, \quad \sigma' = \textstyle\sum_{i \in I} a_i\, \sigma'_i \quad \text{and} \quad s_i \Rightarrow_\partial \sigma'_i \text{ for all } i \in I.$$

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This is bad!

Consider $\sigma = \sum_{n \in \mathbf{N}} \langle \lambda x \, x \rangle^n z$

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Can we tame the combinatorial collapse of term size under parallel reduction?

Bounded chains of consecutive redexes

Definition

Write $s \Rrightarrow_{\langle b \rangle} \sigma'$ if $s \Rrightarrow_{\partial} \sigma'$ so that the chains of immediately nested fired redexes are of length at most b.

Lemma

For all $b \in \mathbf{N}$, there is a function $n \mapsto \mathbf{gb}_b(n)$ such that, if $s \Rrightarrow_{\langle b \rangle} s' + \ldots$ then $\mathbf{s}(s) \leq \mathbf{gb}_b(\mathbf{s}(s'))$.

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But...

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But…

- This is not stable by taking unions of fired redexes.
- Hence, this reduction is not very well behaved (no confluence).

Boundedly nested parallel reduction

Definition

Write $s \Rightarrow_{(b)} \sigma'$ if $s \Rightarrow_{\partial} \sigma'$ so that the nesting tree of fired redexes is of height at most b.

Lemma

For all $b \in \mathbf{N}$, *if* $s \Rightarrow_{(b)} s' + \dots$ *then* $\mathbf{s}\,(s) \leq 4^b \mathbf{s}\,(s')$.

Boundedly nested parallel reduction

Definition

Write $s \Rightarrow_{(b)} \sigma'$ if $s \Rightarrow_{\partial} \sigma'$ so that the nesting tree of fired redexes is of height at most b.

Lemma

For all $b \in \mathbf{N}$*, if* $s \Rightarrow_{(b)} s' + \ldots$ *then* $\mathbf{s}(s) \leq 4^b \mathbf{s}(s')$.

Theorem

$\bigcup_{b \in \mathbf{N}} \widetilde{\Rightarrow_{(\partial)}} b$ *is strongly confluent.*

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Theorem

Restricted to vectors of bounded height, $\widetilde{\Rightarrow_{\partial}}$ is strongly confluent.

β-reduction through Taylor expansion, step 3: problem

Substitution

$$\mathcal{T}(M[N/x]) = \partial_x \mathcal{T}(M) \cdot \mathcal{T}(N)^!$$

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Proof

By induction on M.

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Key lemma

$$\frac{\partial^k \left\langle s \right\rangle \overline{u}}{\partial x^k} \cdot \tau^k = \sum_{k_1 + k_2 = k} \frac{k\,!}{k_1\,!k_2\,!} \left\langle \frac{\partial^{k_1} s}{\partial x^{k_1}} \cdot \tau^{k_1} \right\rangle \frac{\partial^{k_2} \overline{u}}{\partial x^{k_2}} \cdot \tau^{k_2}$$

The uniform case

(Taylor expansions of) pure λ-terms are not stable under $\widetilde{\Rightarrow_\partial}$. But:

Lemma

If $M, N \in \Lambda$ are normalizable and $M \simeq_\partial N$ (i.e. they have a common $\widetilde{\Rightarrow_\partial}^*$-reduct) then $M \simeq_\beta N$.

We use:

- the confluence of $\widetilde{\Rightarrow_\partial}$;
- the stability of NF under $\widetilde{\Rightarrow_\partial}$;
- the injectivity of $\mathcal{T}$ on λ-terms.

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Lemma

If $M, N \in \Lambda$ and $M \simeq_\partial N$ then $\mathcal{BT}(M) = \mathcal{BT}(N)$.

We use:

- the confluence of $\widetilde{\Rightarrow_\partial}$;
- the stability of NF under $\widetilde{\Rightarrow_\partial}$.
- Ehrhard–Regnier's result;
- the injectivity of $\mathcal{T}$ on Böhm trees.

Conclusion

Normalization and Taylor expansion commute

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- can this be generalized to a larger class of terms: "böhmtreeable" terms?
- we claim $\mathcal{BT}(M) := \mathsf{NF}(\mathcal{T}(M))$ (when it is defined): does this coincide with existing notions of (non extensional) Böm trees?
- when is Taylor expansion injective on normal forms? $\rightsquigarrow$ might lead to injectivity results for a class of quantitative denotational models

The end

Thanks for your attention

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Questions?