Context Equivalences and Metrics in Probabilistic λ-Calculi

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Inria informatiques mathématiques

QSLC, Marseille, September 3rd, 2016

Syntax and Operational Semantics of $\Lambda_\oplus$

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$$V \xrightarrow{\mathcal{D}} \mathcal{D}(V) \in \mathbb{R}_{[0,1]} \qquad\qquad \sum \mathcal{D} = \sum_{V} \mathcal{D}(V) \leq 1.$$

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Syntax and Operational Semantics of $\Lambda_\oplus$

$$\frac{}{M \Downarrow \emptyset} \qquad \frac{}{V \Downarrow \{V^1\}} \qquad \frac{M \Downarrow \mathcal{D} \qquad N \Downarrow \mathcal{E}}{M \oplus N \Downarrow \frac{1}{2}\mathcal{D} + \frac{1}{2}\mathcal{E}}$$

$$\frac{M \Downarrow \mathscr{K} \qquad \{P[N/x] \Downarrow \mathscr{E}_P\}_{\lambda x.P \in \mathsf{S}\mathscr{K}}}{MN \Downarrow \sum_{\lambda x.P \in \mathsf{S}\mathscr{K}} \mathscr{K}(\lambda x.P) \cdot \mathscr{E}_P}$$

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$C ::= [\cdot] \;\big|\; \lambda x.C \;\big|\; CM \;\big|\; MC \;\big|\; C \oplus M \;\big|\; M \oplus C$

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- **Context Distance**: $\delta^C(M, N) = \sup_C |\sum [\![C[M]]\!] - \sum [\![C[N]]\!]|$.

Examples

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples

$\lambda x.x$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples

$\Delta\Delta = (\lambda x.xx)(\lambda x.xx)$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Exam

Not Context Equivalent: $C = [\cdot]$.

Context Distance? Consider $C_n = (\lambda x.\ \underbrace{x \ldots x}_{n \text{ times}})[\cdot]$.

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples

$I \oplus \Omega$ vs. I

$I \oplus \Omega$ vs. Ω

Examples

Not Context Equivalent: $C = [\cdot]$.
Context Distance? Cannot Easily Amplify.

$I \oplus \Omega$ vs. I

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$$I \oplus \Omega \qquad \text{vs.} \qquad \Omega$$

$$(\lambda x.I) \oplus (\lambda x.\Omega) \qquad \text{vs.} \qquad \lambda x.I \oplus \Omega$$

Examples

$I \oplus \Omega$ vs. I

Not Context Equivalent in CBV: $C = (\lambda x.x(xI))[\cdot]$
Apparently Context Equivalent in CBN.

$I \oplus \Omega$ vs. Ω

$(\lambda x.I) \oplus (\lambda x.\Omega)$ vs. $\lambda x.I \oplus \Omega$

A Labelled Markov Chain for $\Lambda_\oplus$

Terms

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Terms

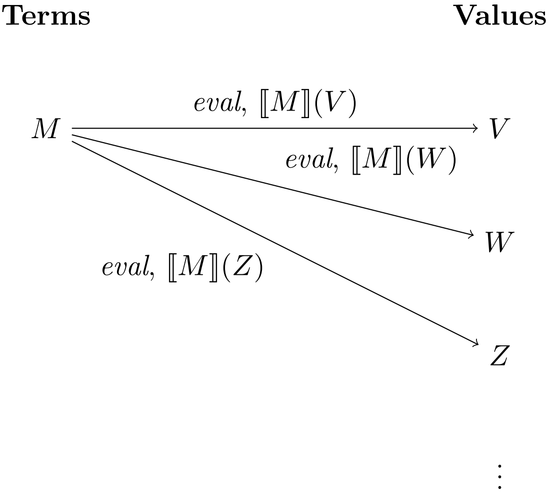
Values

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms **Values**

M

A Labelled Markov Chain for $\Lambda_\oplus$



A Labelled Markov Chain for $\Lambda_\oplus$

Terms	**Values**
	$\lambda x.N$

A Labelled Markov Chain for $\Lambda_\oplus$

$$\begin{tikzcd}
\textbf{Terms} & & \textbf{Values} \\
N\{W/x\} & & \lambda x.N \arrow[ll, "{W,\ 1}"']
\end{tikzcd}$$

Probabilistic Bisimulation Relations

$$\lambda x.M \;\; \mathcal{R} \;\; \lambda x.N$$

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$$\begin{tikzcd}
\lambda x.M \arrow[rrr, bend left, "L"] & \mathcal{R} & \lambda x.N & M\{L/x\}
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$$M \quad \mathcal{R} \quad N$$

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$$\begin{tikzcd}
 & & & \llbracket M \rrbracket \\
M \arrow[rrru, bend left, "eval"] & \mathcal{R} & N &
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 & & & \llbracket M \rrbracket \arrow[dr, bend left] & \\
M \arrow[urrr, bend left, "eval"] & \mathcal{R} & N \arrow[dr, bend right, "eval"'] & & \llbracket M \rrbracket(E) \\
 & & & \llbracket N \rrbracket &
\end{tikzcd}$$

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 & & & \llbracket N \rrbracket \arrow[ur, bend right] &
\end{tikzcd}$$

Bisimilarity vs. Context Equivalence

- **Bisimilarity**: the union $\sim$ of all bisimulation relations.
- Is it that $\sim$ is included in $\equiv$? How to prove it?
- Natural strategy: is $\sim$ a congruence?
  - If this is the case:

$$\begin{aligned} M \sim N &\implies C[M] \sim C[N] \implies \sum [\![C[M]]\!] = \sum [\![C[N]]\!] \\ &\implies M \equiv N. \end{aligned}$$

  - This is a necessary sanity check anyway.
- The naïve proof by induction **fails**, due to application: from $M \sim N$, one cannot directly conclude that $LM \sim LN$.

Howe's Technique

$\mathcal{R}$ $\mathcal{R}^H$

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$$\begin{tikzcd}
\mathcal{R} \arrow[rr, bend left, "\subseteq"] & & \mathcal{R}^H
\end{tikzcd}$$

Howe's Technique

$\mathcal{R}^H$ is a **Congruence** whenever $\mathcal{R}$ is an equivalence

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Howe's Technique

$\sim^H$ is a **Congruence**

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$\sim^H$ is a **Congruence**

$$\begin{tikzcd}
\sim \arrow[r, bend left, "\subseteq"] & \sim^H \arrow[l, bend left, "\supseteq"]
\end{tikzcd}$$

Key Lemma

Our Neighborhood

- Λ, where we observe **convergence**

	$\sim \subseteq \equiv$	$\equiv \subseteq \sim$
CBN	✓	✓
CBV	✓	✓

[Abramsky1990, Howe1993]

- $\Lambda_{\oplus}$ with nondeterministic semantics, where we observe **convergence**, in its **may** or **must** flavors.

	$\sim \subseteq \equiv$	$\equiv \subseteq \sim$
CBN	✓	×
CBV	✓	×

[Ong1993, Lassen1998]

The Probabilistic Case

- $\Lambda_{\oplus}$ with probabilistic semantics.

	$\sim \subseteq \equiv$	$\equiv \subseteq \sim$
CBN	✓	✗
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- Counterexample for CBN: $(\lambda x.I) \oplus (\lambda x.\Omega) \not\sim \lambda x.I \oplus \Omega$
- **Where** these discrepancies come from?

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- From **testing**!

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- From **testing**!
- Bisimulation can be characterized by testing equivalence as follows:

Calculus	**Testing**
Λ	$T ::= \omega \mid a \cdot T$
$P\Lambda_{\oplus}$	$T ::= \omega \mid a \cdot T \mid \langle T, T \rangle$
$N\Lambda_{\oplus}$	$T ::= \omega \mid a \cdot T \mid \wedge_{i \in I} T_i \mid \ldots$

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	$\precsim\ \subseteq\ \leq$	$\leq\ \subseteq\ \precsim$
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$$T ::= \omega \;\Big|\; a \cdot T \;\Big|\; \langle T, T \rangle \;\Big|\; T \vee T$$

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- Full abstraction can be recovered if endowing $\Lambda_\oplus$ with parallel disjunction [CDLSV2015].

	$\precsim \subseteq \leq$	$\leq \subseteq \precsim$
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$$\frac{}{\Gamma, x \vdash x} \qquad \frac{x, \Gamma \vdash M}{\Gamma \vdash \lambda x.M} \qquad \frac{\Gamma \vdash M \qquad \Delta \vdash N}{\Gamma, \Delta \vdash MN} \qquad \frac{\Gamma \vdash M \qquad \Gamma \vdash N}{\Gamma \vdash M \oplus N}$$

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  - The maximum distance induced by traces, i.e., sequences of actions: $\delta^t(M, N) = \sup_{\mathsf{T}} |Pr(M, \mathsf{T}) - Pr(N, \mathsf{T})|$.

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- **Soundness and Completeness Results:**

$\delta^b \leq \delta^c$	$\delta^c \leq \delta^b$	$\delta^t \leq \delta^c$	$\delta^c \leq \delta^t$
✓	✗	✓	✓

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- **Example**: $\delta^t(I, I \oplus \Omega) = \delta^t(I \oplus \Omega, \Omega) = \frac{1}{2}$.

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- N
  δ
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- **Examples**: $\delta^t(!(I \oplus \Omega), !\Omega) = \frac{1}{2}$ $\qquad \delta^t(!(I \oplus \Omega), !I) = 1$.
- **Trivialisation** does not hold in general, but becomes true in *strongly normalising* fragments or in presence of *parellel disjuction*.

Open Problems

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  - We have a preliminary work on characterizing it as trace equivalence in a λ-calculus for polynomial time, called RSLR [CDL2015].

C

$\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ (where both $\mathcal{D}_n$ and $\mathcal{E}_n$ are distributions on binary strings) are said to be *computationally indistinguishable* iff for every PPT algorithm $\mathcal{A}$ the following quantity is a negligible function of $n \in \mathbb{N}$:

$$\left|\Pr_{x\leftarrow\mathcal{D}_n}(\mathcal{A}(x,1^n)=\epsilon)-\Pr_{x\leftarrow\mathcal{E}_n}(\mathcal{A}(x,1^n)=\epsilon)\right|.$$

ween two

$]\!]$ and $[\![N]\!]$

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- Higher-order computational indistinguishability?

Thank You!

Questions?