The Resource Lambda Calculus
What's there and what's missing?

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The λ-calculus Λ — its syntax and semantics

Syntactic Results

Böhm Theorem

- Separability for β-nfs M, N

Solvability

- Interaction with the environment

Böhm Trees $BT(M)$

- Trees representing executions

Observational Theories

- Program equivalence $M \cong N$

Characterizations based on trees

$M \cong N \iff BT(M) =_{\eta^\infty} BT(N)$

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Semantic Results

Abstract notions of model

- $[D \Rightarrow D] \triangleleft D$ in CCC
- Combinatory Algebras

Classes of Models

- Continuous Semantics
- stable, strongly stable, games and relational semantics.

Approximation Theorem

- $[\![M]\!] = \bigvee_{a \in App(BT(M))} [\![a]\!]$

Full Abstraction

$M \cong N \iff \mathcal{D}_\infty \models M = N$

The resource calculus Λ^r

Λ^r extends the notion of the λ-calculus application along two directions:

$$MN$$

1. a term is applied to a multiset of resources, called bag
2. the resources can be reusable or linear

Tranquilli. Nets Between Determinism and Non-determinism. PhD Thesis (2009)

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Just another syntax for the differential λ-calculus:

$$M[L_1, \ldots, L_\ell, N_1^!, \ldots, N_n^!] \cong \big(\mathrm{D}^\ell(M) \cdot (L_1, \ldots, L_\ell)\big)(\Sigma_i N_i)$$

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Ehrhard, Regnier. The differential lambda-calculus. Theor. Comput. Sci. 309(1-3): 1-41 (2003)

The syntax of Λ^r

There are three syntactic categories:

- terms are in functional positions,
- bags are in argument position and represent multisets of linear and reusable resources,
- sums of terms are results of the computation.

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\begin{array}{lll}
M, N, L & := x \mid \lambda x.M \mid MP & \text{terms} \\
P, Q, R & := [] \mid [M] \mid [M^!] \mid P \uplus Q & \text{bags} \\
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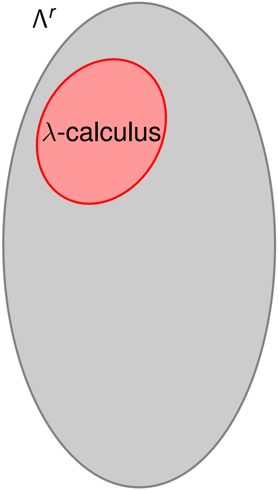

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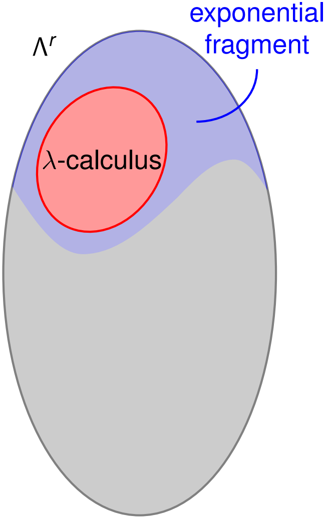

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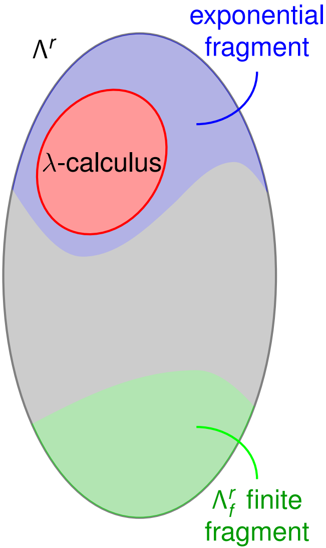
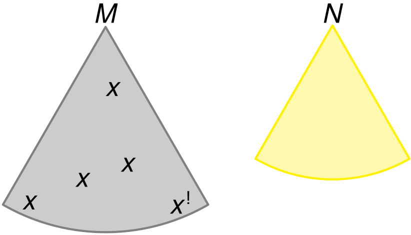


Two kind of substitutions

Usual and Linear Substitution

Two kinds of resources $\Rightarrow$ two kinds of substitution:

- $M\{N/x\}$: usual capture free substitution,
- $M\langle N/x\rangle$: *linear* substitution, N is substituted for **exactly one linear** occurrence of x in M.



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substitute all occurrences!

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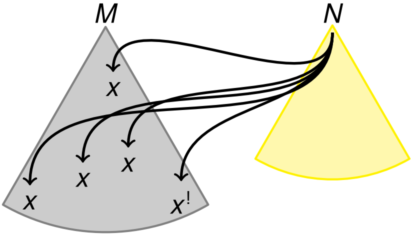

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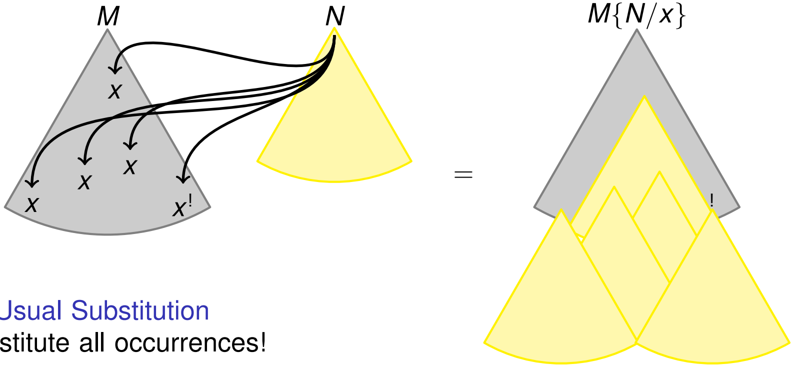

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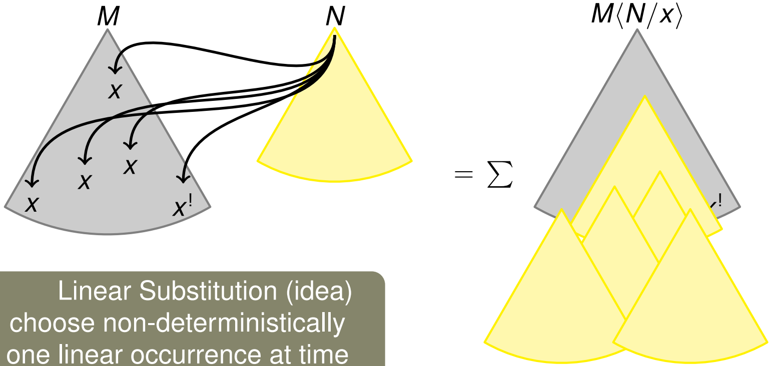

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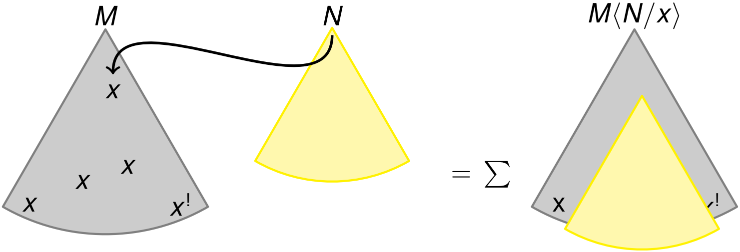

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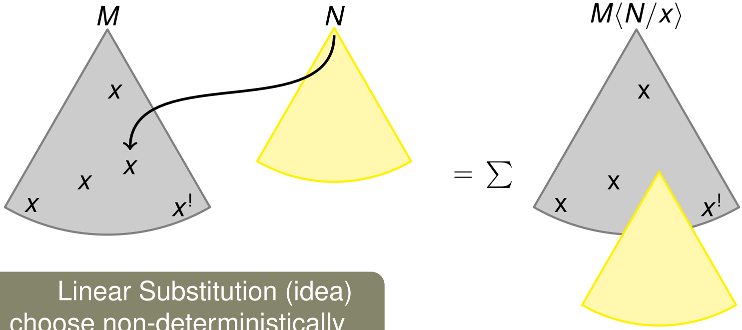

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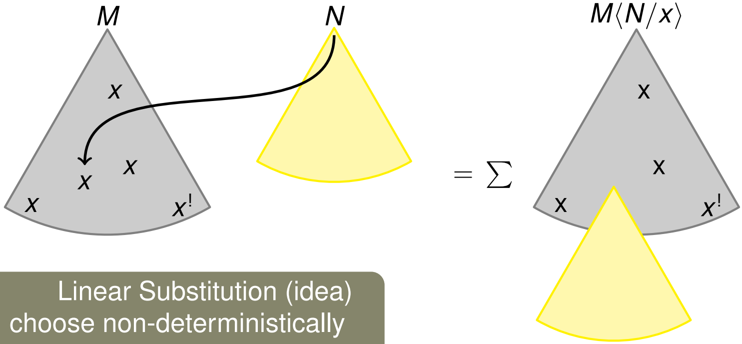

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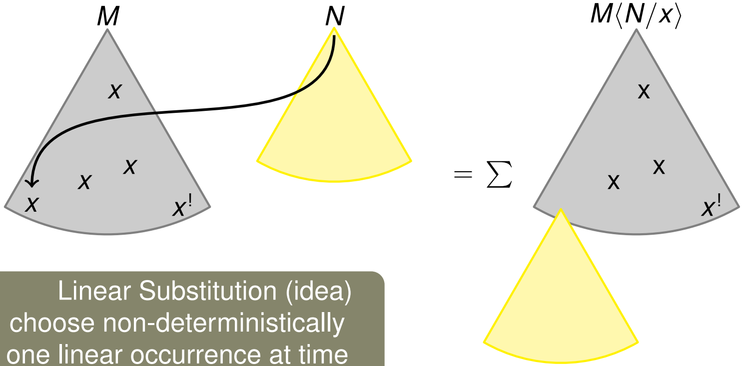

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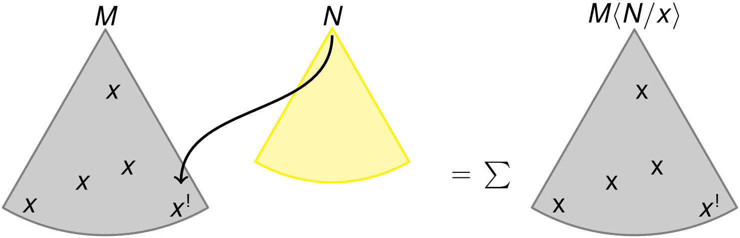

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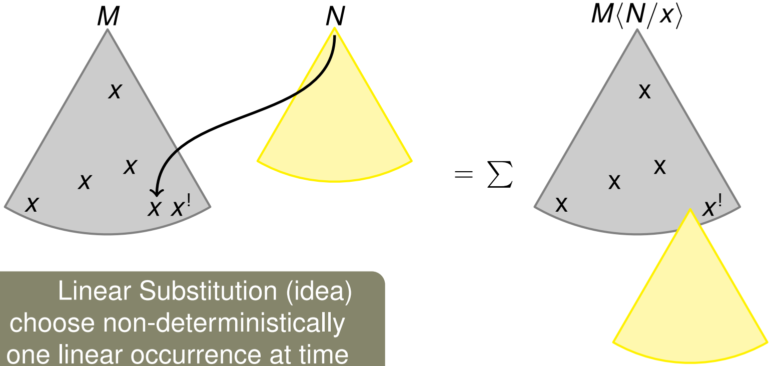

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The operational semantics of Λ^r

β- and η- reductions

The β-reduction:

$$(\lambda x.M)[L_1, \ldots, L_\ell, N_1^!, \ldots, N_n^!] \xrightarrow{\beta} M\langle L_1/x\rangle \cdots \langle L_\ell/x\rangle \{\Sigma_i N_i/x\}$$

The η-reduction:

$$\lambda x.M[x^!] \xrightarrow{\eta} M, \text{ where } x \notin \mathrm{FV}(M)$$

Theorem (Pagani, Tranquilli APLAS'09)

- *A Standardization Theorem holds ($\xrightarrow{\beta *} := \xrightarrow{\mathtt{o}*}\xrightarrow{\mathtt{i}*}$)*
- *The reductions of Λ^r are confluent.*

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Pagani, Tranquilli. Parallel Reduction in Resource Lambda-Calculus. APLAS 2009: 226-242

Syntactic Results

Solvability

A closed term M is

- may-solvable if there are closed bags $\vec{P}$ such that $M\vec{P} \stackrel{\beta*}{\to} \mathbf{I} + \mathbb{N}$ for some $\mathbb{N}$,
- must-solvable if there are closed bags $\vec{P}$ such that $M\vec{P} \stackrel{\beta*}{\to} \mathbf{I} + \cdots + \mathbf{I}$.

Pagani, Ronchi Della Rocca: Linearity, Non-determinism and Solvability. Fundam. Inform. 103(1-4): 173-202 (2010)

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Characterization of May-Solvability (Pagani and Ronchi 2010)

- M is may-solvable,
- M reduces outermost to $O + \mathbb{N}$ where O is an outer-nf
- M is typable in System m

Open Problem 1: characterization of must solvability.

Pagani, Ronchi Della Rocca: Linearity, Non-determinism and Solvability. Fundam. Inform. 103(1-4): 173-202 (2010)

Separability

Obviously $M + M$ and M are *inseparable* $\Rightarrow$ we consider an idempotent sum.

There are $\beta\eta$-distinct normal forms that **cannot** be separated

$$\mathbb{M} := \lambda z.x[] + \lambda z.x[z, z^!] \qquad \mathbb{N} := \lambda z.x[z^!]$$

- We need a resource sensitive version of Böhm trees,

- We need a clever version of η-equivalence.

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  The Taylor Expansion of a regular λ-term:

$$\mathcal{T}(MN) = \sum_{n=0}^{\infty} \frac{1}{n!} \big(\mathrm{D}^{n}(M) \cdot (\underbrace{N, \dots, N}_{n \text{ times}})\big)(0)$$

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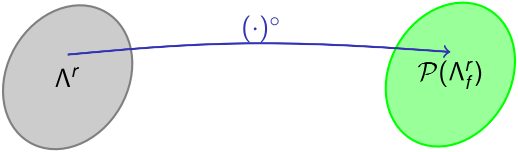
The Taylor Expansion



The Taylor Expansion $\mathcal{T}(M)$ of a term M is obtained by expanding the bangs:

$$[L_1, \dots, L_\ell, N_1^!, \dots, N_n^!] \mapsto \bigcup_{k_1,\dots,k_n=0}^{\infty} [L_1, \dots, L_\ell, \underbrace{N_1, \dots, N_1}_{k_1 \text{ times}}, \dots, \underbrace{N_n, \dots, N_n}_{k_n \text{ times}}]$$

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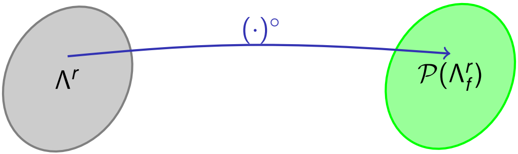

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Example

- $\mathcal{T}(\mathbb{M}) := \mathcal{T}(\lambda z.x[z^!]) = \{\lambda z.x[z^n] : n \in \mathrm{Nat}\}$,
- $\mathcal{T}(\mathbb{N}) := \mathcal{T}(\lambda z.x[] + \lambda z.x[z,z^!]) = \{\lambda z.x[]\} \cup \{\lambda z.x[z^{n+1}] : n \in \mathrm{Nat}\}$.

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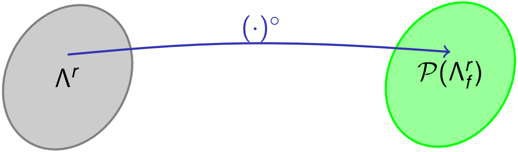

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Theorem (Ehrhard-Regnier)

- *$NF(\mathcal{T}(M)) \neq \emptyset$ if and only if M is may-solvable.*
- *For λ-terms $M \equiv_\tau N$ if and only if $BT(M) = BT(N)$*

Resource Böhm's Theorem

Theorem (Resource Böhm's Theorem, Manzonetto-Pagani 2011)

Let $\mathbb{M}, \mathbb{N}$ closed sums of terms in β-nf. If $\mathbb{M} \not\equiv_{\eta\tau} \mathbb{N}$ then $\exists$ closed bags $\vec{P}$ s.t.:

$$\mathbb{M}\vec{P} \stackrel{\beta*}{\to} \mathbf{I} \qquad \mathbb{N}\vec{P} \stackrel{\beta*}{\to} 0 \qquad \textit{(or vice versa).}$$

Manzonetto, Pagani: Böhm's Theorem for Resource Lambda Calculus through Taylor Expansion. TLCA 2011: 153-168

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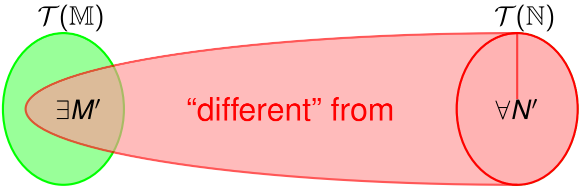
Difficult: one needs to separate a finite term from infinitely many different finite terms!



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Observational Preorder/Equivalence

Given closed $\mathbb{M}, \mathbb{N}$ we have $\mathbb{M} \sqsubseteq \mathbb{N}$ whenever

$$\forall \text{ closed } \vec{P}, \mathbb{M}\vec{P} \text{ is may-solvable } \Rightarrow \mathbb{N}\vec{P} \text{ is may-solvable } .$$

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Open Problem 2:
Is there a characterization of $\cong$ in terms of Taylor Expansion + η-equivalence?

Attention! Much more difficult to handle η on arbitrary Taylor Expansions:

$$M \stackrel{\beta*}{\to} \lambda x.y[] + \lambda x.y[x] + \lambda x.y[x,x] + M' \to_{lim} \sum_{n=0}^{\infty} \lambda x.y[x^n] \equiv_\eta y$$

Manzonetto, Ruoppolo: Relational Graph Models, Taylor Expansion and Extensionality. Electr. Notes Theor. Comput. Sci. 308: 245-272 (2014)

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$$\text{is } \lambda x.y[] \in \mathcal{T}(M) \text{ part of a } \eta\text{-redex?}$$

η-reduction is not a local operation: in general $\lambda x.y[x^n] \not\xrightarrow{\eta} y$

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Denotational Semantics

Cartesian Differential Categories

Cartesian differential operator:

$$\frac{f : A \to B}{D(f) : \underline{A} \times A \to B}$$

Satisfying:

D1. $D(f+g) = D(f) + D(g)$ and $D(0) = 0$

D2. $\langle h+k, v\rangle; D(f) = \langle h, v\rangle; D(f) + \langle k, v\rangle; D(f)$ and $\langle 0, v\rangle; D(f) = 0$

D3. $D(Id) = \pi_1$, $D(\pi_1) = \pi_1; \pi_1$ and $D(\pi_2) = \pi_1; \pi_2$

D4. $D(\langle f, g\rangle) = \langle D(f), D(g)\rangle$

D5. $D(g; f) = \langle D(g), \pi_2; g\rangle; D(f)$

D6. $\langle\langle g, 0\rangle, \langle h, k\rangle\rangle; D(D(f)) = \langle g, k\rangle; D(f)$

D7. $\langle\langle 0, h\rangle, \langle g, k\rangle\rangle; D(D(f)) = \langle\langle 0, g\rangle, \langle h, k\rangle\rangle; D(D(f))$

R. F. Blute, J. R. B. Cockett, R. A. G. Seely, Cartesian differential categories, Theory and Applications of Categories 22 (2009) 622–672.

Cartesian Closed Differential Categories

Cartesian closed structure + new axiom:

$$D(\Lambda(f)) = \Lambda(\langle \pi_1 \times 0_A, \pi_2 \times Id_A \rangle; D(f))$$

Linear reflexive object:

$$\begin{tikzcd} {[D \Rightarrow D]} \arrow[rr, bend left, "\mathrm{App}"] \arrow[rr, phantom, "\lhd"] & & D \arrow[ll, bend left, "\mathrm{Lam}"] \end{tikzcd}$$ where App and Lam are linear morphisms.

Examples:

- The relational semantics [Bucciarelli et Al. CSL'07]
- The weighted relational semantics [Laird et Al. LICS'13]
- non-deterministic games [Laird et Al. ICALP'11]

Manzonetto. What is a categorical model of the differential and the resource λ-calculi? Math. Struct. in Comp. Sci. 22(3): 451-520 (2012)

The Relational Semantics

The cartesian closed category **MRel**:

objects: sets A, B, ...

morphisms: $A \to B$ are relations from $\mathcal{M}_f(A)$ to B

Girard: Normal functors, power series and λ-calculus. Ann. Pure Appl. Logic 37(2): 129-177 (1988)

The Relational Semantics (Quantitative)

The cartesian closed category **MRel**:

objects: sets A, B, $\ldots$

morphisms: $A \to B$ are relations from $\mathcal{M}_f(A)$ to B

Finite multisets comonad $\mathcal{M}_f(-)$

Number of calls to an input of type A to produce an output of type B.

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The model $\mathcal{D}$

$$D_0 = \emptyset \qquad D_{n+1} = M_f(D_n)^{(\omega)} \qquad \mathcal{D} = \cup_{n\in\omega} D_n$$

$$(m_1, m_2, m_3, \ldots) \in \mathcal{D} \iff (m_1, (m_2, m_3, \ldots)) \in M_f(\mathcal{D}) \times \mathcal{D} = [\mathcal{D} \Rightarrow \mathcal{D}]$$

The model has a special element $\varepsilon = ([], [], \ldots, [], \ldots) = ([], \varepsilon)$.

Bucciarelli, Ehrhard, Manzonetto: Not Enough Points Is Enough. CSL 2007: 298-312

The structure of $\mathcal{D}$

Theorem (Approximation Theorem for TE)

For all closed $M \in \Lambda^r$, we have:

$$\sigma \in [\![M]\!] \iff \exists t \in \mathcal{T}(M) \text{ such that } \sigma \in [\![t]\!]$$

That is $[\![M]\!] = \bigcup_{t \in \mathcal{T}(M)} [\![t]\!]$.

$\mathcal{D}$ was a good candidate to be a fully abstract model of Λ^r...

- $\mathcal{D}$ is fully abstract for Λ [Manzonetto MFCS'09],
- $\mathcal{D}$ is fully abstract for Λ^r with convergency tests. [Bucciarelli et Al. CSL'11]

Manzonetto: A General Class of Models of H*. MFCS 2009: 574-586

Bucciarelli, Carraro, Ehrhard, Manzonetto: Full Abstraction for Resource Calculus with Tests. CSL 2011: 97-111

Not a Fully Abstract Model of Λ^r

The following term A is such that $\mathbf{I} \sqsubseteq A$:

$$A \xrightarrow{\beta}_{lim} \sum_{n\geq 1} B_n \qquad \text{with} \qquad B_n = \lambda v_1 \dots v_n x.x[\mathbf{I}[v_1^!] \cdots [v_n^!]]$$

$\forall$ closed $\vec{P}$. $\mathbf{I}\vec{P}$ is may-solvable $\Rightarrow A\vec{P} \xrightarrow{\beta *} \lambda x.x[\mathbf{I}P_1 \cdots P_n] + \mathbb{A}'$ is may-solvable.

However: $([\varepsilon], \varepsilon) \in [\![\mathbf{I}]\!] - [\![A]\!]$, so $\mathcal{D} \models \mathbf{I} \not\sqsubseteq A$

Open Problems:

3. Is there a FA model in the relational semantics? In other semantics?
4. Is it possible to characterize the differential theory of $\mathcal{D}$?

Breuvart: The Resource Lambda Calculus Is Short-Sighted in Its Relational Model. TLCA 2013: 93-108

Conclusions

We revised the known results about Λ^r

- Confluence,
- Solvability,
- Böhm Theorem,
- Taylor Expansions vs extensionality,
- Observational Theory.

Some Open Problems:

1. characterization of must solvability.
2. characterization of $\cong$ in terms of an extensional Taylor Expansion
3. Is there a FA model in the relational semantics? In other semantics?
4. Is it possible to characterize the differential theory of $\mathcal{D}$?
5. What is an algebraic model of resource calculus?