# All About That Base

Qualitative to Quantitative Denotational Models

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# Aims

- Give a categorical model of higher-order computation with quantities.
- A general way of moving from existing, qualitative models, to quantitative ones ...
- ... which captures existing examples (non-deterministic games [Harmer and McCusker], probabilistic games [Danos and Harmer], slot games [Ghica]).

# Enriched Categories

We may represent the two level structure we want to capture (computation, quantities) using *enriched categories*:
Let $\mathcal{V}$ be a monoidal category (the *base*).Recall that a $\mathcal{V}$-category $\mathcal{C}$ is given by:

- A set $Obj_{\mathcal{C}}$
- A $\mathcal{V}$-object $\mathcal{C}(A, B)$ for each $A, B \in Obj_{\mathcal{C}}$.
- $\mathcal{V}$-morphisms $\mathrm{id}_A : I \to \mathcal{C}(A, A)$ and $comp_{A,B,C} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ satisfying the relevant diagrams.

# Constructing Differential Categories, Deconstructing Games

[L., McCusker Manzonetto, I&C '13] Start with any symmetric monoidal category, $\mathcal{C}$:

1. Freely enrich over sup-lattices (i.e. morphisms are *sets* of $\mathcal{C}$-morphisms), to get a commutative-monoid-enriched category.
2. *Complete* with countable, distributive *biproducts*
3. Take the *Karoubi envelope* (free idempotent splitting) — giving *symmetric tensor powers* by splitting the idempotents

We can then construct the *cofree commutative comonoid* by Lafont's construction, giving a differential closed category.

# A blueprint for constructing quantitative models?

- Starting with $\mathcal{C}$ as the category with one-object and one morphism, we get the category of relations, with the multiset exponential.
- Starting with $\mathcal{C}$ as a simple category of games (essentially, the free SMCC over 1 object), we get a HO-style, fully complete model of "differential PCF".

Can we generalize this? The problematic step is free sup-lattice enrichment:

- Applying this to e.g. history sensitive strategies does not give us non-deterministic strategies.
- How do we enrich over other kinds of quantities — e.g. natural numbers, probabilities, dioids?

# $R$-weighted Relations

[L., Manzonetto,McCusker,Pagani - LICS '13] For any *continuous semiring* $\mathcal{R}$:

- Construction of a Lafont category (model of ILL with free exponential) in which objects are sets and morphisms from $A$ to $B$ are $A \times B$ matrices.
- Models of PCF with a sum operator and weights from $\mathcal{R}$.
- Key property: Computational Adequacy — denotation of a program given by the weighted sum of its reduction paths. **Not** fully abstract.

Can we generalize this?

- To models with richer structure (e.g. games) — interpret computational effects, prove full abstraction.
- To complete semirings (can't enrich over continuous monoids, and some interesting examples are not continuous).

# Lafont Categories With Biproducts

[L. LICS'16] A categorical semantics for quantitative higher-order computation: *Lafont categories with biproducts* — in which we define:

1. a parameterised uniform fixed point operator, using principles of axiomatic domain theory.
2. a computationally adequate semantics of $\mathrm{PCF}^+$ with weights from the *internal semiring* — proved using a resource $\lambda$-calculus with *nested finite multiset* resources.

# Biproducts

A category $\mathcal{C}$ has (finite) biproducts if it has (finite) products and coproducts, and these are naturally isomorphic:

- The category of sets and functions does not have biproducts.
- The category of $\mathbb{F}$-vector spaces always has finite biproducts (the direct sum) but never has infinite biproducts.
- The category of sets and relations has all biproducts (disjoint unions).

Note that biproducts in a symmetric monoidal closed category are distributive: $A \otimes \bigoplus_{i \in I} B_i \cong \bigoplus_{i \in I} (A \otimes B_i)$.

# From Biproducts to Complete Monoid Enrichment

In a category with biproducts we may define the sum of a family of morphisms $\{f_i : A \to B\}_{i \in I}$:

$$\Sigma_{i \in I} f_i = \langle f_i \mid i \in I \rangle ; [\mathrm{id}_B \mid i \in I] = \langle \mathrm{id}_A \mid i \in I \rangle ; [f_i \mid i \in I]$$

This is a complete monoid — an indexed sum $\Sigma$ such that:

- If $I$ is partitioned into $\{I_j \mid j \in J\}$ then $\Sigma_{i \in I} a_i = \Sigma_{j \in J} \Sigma_{i \in I_j} a_i$.
- If $I = \{j\}$ then $\Sigma_{i \in I} a_i = a_j$.

$\mathcal{C}$ is enriched over the category of complete monoids.

# Symmetric Monoidal Categories and Complete Semirings

- In any (symmetric) monoidal category $(\mathcal{C}, \otimes, I)$ we may define "scalar multiplication" of $g : A \to B$ by $f : I \to I$

$$f \cdot g : A \cong I \otimes A \xrightarrow{f \otimes g} I \otimes B \cong B$$

- If $\mathcal{C}$ (and $\otimes$) is complete-monoid-enriched, scalar multiplication distributes over the sum, so $\mathcal{C}(I, I)$ is a (commutative) complete semiring — the "internal semiring" $\mathcal{R}_C = (\mathcal{C}(I, I), \Sigma, \cdot, \mathrm{id}_I)$.
- $\mathcal{C}$ is enriched over the category of $\mathcal{R}_C$-modules.

# From Complete Monoid Enrichment to Biproducts

For $\mathcal{C}$ complete monoid enriched, the biproduct completion $\mathcal{C}^{\Pi}$ has:

- Objects — indexed families of objects of $\mathcal{C}$.
- Morphisms from $\{A_i\}_{i \in I}$ to $\{B_j\}_{j \in J}$ — $I \times J$ "matrices" of morphisms; $\{f_{ij} : A_i \to B_j\}_{ij \in I \times J}$

Composition is by "matrix multiplication":

$$\{f_{ij}\}_{ij \in I \times J} ; \{g_{jk}\}_{jk \in J \times K} = \{\Sigma_{j \in J} f_{ij} ; g_{jk}\}_{ik \in I \times K}$$

Since any complete, commutative semiring $\mathcal{R}$ is a one-object complete-monoid-enriched SMCC, $\mathcal{R}^{\Pi}$ is a SMCC with biproducts — a.k.a.

- The category of sets and $\mathcal{R}$-weighted relations.
- The category of free $\mathcal{R}$-modules and their homomorphisms.

# Lafont Categories

A SMCC with (bi)products $\mathcal{C}$ is a Lafont Category if the forgetful functor into $\mathcal{C}$ from its category of commutative comonoids has a right adjoint.

In other words, for every object $A$ in $\mathcal{C}$ there is:

- A commutative comonoid $(!A, \delta_A : !A \to !A \otimes !A, \epsilon_A : !A \to I)$
- which is "cofree": there is a morphism $\mathsf{der}_A : !A \to A$ giving a natural isomorphism between morphisms into $A$ and comonoid morphisms into $!A$.

Resolving this (monoidal) adjunction gives a (monoidal) comonad $! : \mathcal{C} \to \mathcal{C}$, and thus a (cartesian closed) co-Kleisli category $\mathcal{C}_!$ .

# Cofree commutative comonoids in $\mathcal{R}^\Pi$

**Proposition** $\mathcal{R}^\Pi$ is a Lafont category.

For any set $S$, $!S$ is the set $\mathcal{M}_*(S)$ of finite multisets over $S$, with:

- $\delta_{XYZ} = 1$ if $X = Y \uplus Z$ and 0 otherwise.
- $\epsilon_{X*} = 1$ if $X = [\_]$ and 0 otherwise.
- $\mathrm{der}_{XY} = 1$ if $X = [Y]$ and 0 otherwise.

For any $f :!S \to T$, define the comonoid morphism
$f^\dagger_{X,[y_1,\ldots,y_n]} :!S \to !T = \Sigma\{f_{X_1,y_1} \cdot \ldots \cdot f_{X_n,y_n} \mid X = X_1 + \ldots + X_n\}$

# Fixed Point Operators

To interpret recursively defined functions we require a fixed point operator for $\mathcal{C}_!$ — a map sending $f : !A \rightarrow A$ to $\mathrm{fix}(f) : !1 \rightarrow A$ such that $\mathrm{fix}(f) = \mathrm{fix}(f)^\dagger; f$.
It is uniform in ! if $f; h = !h; g$ implies $\mathrm{fix}(g) = \mathrm{fix}(f); h$.

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How can we obtain such a fixed point?

- CPO enrichment of $\mathcal{C}$? — Only definable if the internal semiring is continuous.
- Trace operator on $\mathcal{C}$? — Only uniform (and adequate) if the internal semiring is idempotent.
- A Bifree Algebra for the exponential ?

# Fixed Points from Bifree Algebras

A bifree algebra for $! : \mathcal{C} \to \mathcal{C}$ is an object $A$ with an isomorphism $\alpha : !A \cong A$ such that:

- $\alpha : !A \to A$ is an initial algebra for !.
- $\alpha^{-1} : A \to !A$ is a final coalgebra for !.

# Fixed Points from Bifree Algebras

**Proposition**[After Freyd (1990) and Simpson and Plotkin (200)] If $! : \mathcal{C} \to \mathcal{C}$ has a bifree algebra $\Psi$ then $\mathcal{C}_!$ has a unique uniform fixed point operator.

- $\mathrm{id}_{!1}^{\dagger} :!1 \to !!1$ has a unique anamorphism $\infty :!1 \to \Psi$ such that $\infty; \psi^{-1} = \mathrm{id}_{!1}^{\dagger}; !\infty = \infty^{\dagger}$.
- Any $f :!A \to A$ has a unique catamorphism $(\![f]\!) : \Psi \to A$ such that $\psi; (\![f]\!) = !(\![f]\!); f$. Let $\mathrm{fix}(f) :!1 \to A = \infty; (\![f]\!)$
- $\mathrm{fix}(f)^{\dagger}; f = (\infty; (\![f]\!))^{\dagger}; f = \infty^{\dagger}; !(\![f]\!); f = \infty; \psi^{-1}; \psi; (\![f]\!) = \infty; (\![f) = \mathrm{fix}(f)$.
- Uniformity and uniqueness follow from uniqueness of catamorphisms and anamorphisms.

# The Bifree Algebra of Nested Finite Multisets

**Theorem** If $\mathcal{C}$ is a Lafont category with biproducts then $\mathcal{C}_!$ has a uniform fixed point operator.

Observe that the functor from $\mathcal{R}^{\prod}_{\mathcal{C}}$ to $\mathcal{C}$ sending $S$ to $\bigoplus_{a \in S} I$ preserves cofree exponentials — i.e. $! \bigoplus_{a \in S} I \cong \bigoplus_{a \in \mathcal{M}_*(S)} I$

- Let $\mathbb{M}$ be the set of finite nested multisets — i.e. $\mathbb{M} = \bigcup_{i \in \mathbb{N}} \mathbb{M}_i$, where $\mathbb{M}_0 = \varnothing$ and $\mathbb{M}_{i+1} = \mathcal{M}_*(\mathbb{M}_i)$. Then $\mathcal{M}_*(\mathbb{M}) = \mathbb{M}$.
- Hence $\Psi = \bigoplus_{i \in \mathbb{M}} I = \bigoplus_{i \in \mathcal{M}_*(\mathbb{M})} I$ and so $!\Psi \cong \Psi$ in $\mathcal{C}$.

We show that this isomorphism is a bifree algebra for $! : \mathcal{C} \to \mathcal{C}$.

# $R$-weighted $\lambda$-terms

Extend an applied $\lambda$-calculus (PCF) with:

- Erratic choice:

$$\frac{\Gamma \vdash M : \mathtt{nat} \qquad \Gamma \vdash N : \mathtt{nat}}{\Gamma \vdash M \text{ or } N : \mathtt{nat}}$$

- "Scalar Weights" from a complete semiring $\mathcal{R}$:

$$\frac{\Gamma \vdash M : \mathtt{nat}}{\Gamma \vdash \underline{a}.M : \mathtt{nat}} \, a \in \mathcal{R}$$

# Operational Semantics

Define a labelled transition system in which

- states are programs of $\mathrm{PCF}^R$ (closed terms of ground type).
- labels are elements of $\{l, r\}^* \times |\mathcal{R}|$.
- actions are as follows:

$$
\begin{array}{lcllcl}
E[M \text{ or } N] & \xrightarrow{l,1} & E[M] & \qquad E[M \text{ or } N] & \xrightarrow{r,1} & E[N] \\
E[\underline{a}.M] & \xrightarrow{\varepsilon,a} & E[M] & \qquad E[M] & \xrightarrow{\varepsilon,1} & E[M']
\end{array}
$$

where $M \longrightarrow M'$ is a PCF reduction.

# Path Weights

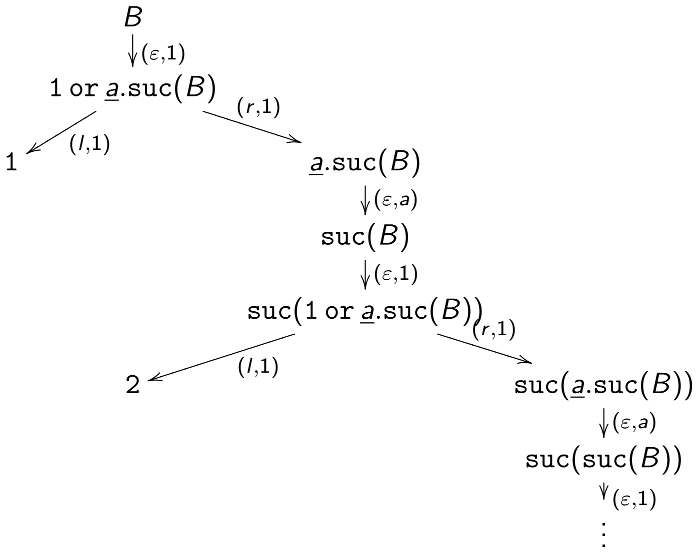
For closed terms $M, N$ and a path $s \in \{l, r\}^*$, define a weight in $\mathcal{R}$:

- $w_s(M, N) = a_1 \cdot \ldots \cdot a_n$ if there exists a reduction $M \xrightarrow{u_1, a_1} \ldots \xrightarrow{u_n, a_n} N$ such that $s = u_1 \cdot \ldots \cdot u_n$.
- $w_s(M, N) = 0$, otherwise.

We evaluate $M$ at $n$ by taking the sum in $\mathcal{R}$ of path-weights: $W(M, n) = \Sigma_{s \in \{l, r\}^*} w_s(M, \mathtt{n})$.
Note that if $M$ is strongly convergent (has no infinite reduction paths) this is a finite sum by König's Lemma.

# Example

Let $B = \mu x.1 \text{ or } \underline{a}.\mathtt{suc}(x)$:




$W(M, n) = a^{n-1}$ for $n > 0$.

# Continuous Semirings

There are many computationally interesting examples of complete semirings e.g.:

- complete lattices (security),
- positive reals, $(\mathcal{R}_+ \cup \infty, \Sigma, \times, 1)$ (probability)
- tropical semirings —e.g.$(\mathbb{N} \cup \infty, \mathtt{max}, +, 0)$ (longest path).

The above are continuous semirings — they can be ordered so that sum and product are continuous.

# Non-continuous Semirings

Let $(|\mathcal{R}|, +, \cdot, 1)$ be any commutative semiring without (necessarily) additive identity. Define a complete semiring $\mathcal{R}_0^\infty = (|\mathcal{R}| \uplus \{0, \infty\}, \Sigma, \cdot, 1)$, where:

- $\Sigma_{i \in I} a_i = 0$ if $I_0^C \triangleq \{i \in I \mid a_i \neq 0\} = \varnothing$
- $\Sigma_{i \in I} a_i = \Sigma_{i \in I_0^C} a_i$, if $0 < |I_0^C| < \infty$ and $\{i \in I \mid a_i = \infty\} = \varnothing$.
- $\Sigma_{i \in I} a_i = \infty$, otherwise.

and $a.\infty = \infty.a = 0$ if $a = 0$; $a.\infty = \infty.a = \infty$, otherwise.
This is not continuous in general — e.g. if $\mathcal{R}$ is finite.

# Nested multiset approximants

To prove computational adequacy, we represent fixed points as sums of approximants, indexed over $\mathbb{M}$:

- We show that $\infty : I \to \Psi = \Sigma_{X \in \mathbb{M}} \iota_X$ (where $\iota_X : I \to \bigoplus_{X \in \mathbb{M}} I$ is the $X$th injection.
- Let $f^X : I \to !A = \iota_X ; (\![f]\!)$.

Then $\mathrm{fix}(f) = \Sigma_{X \in \mathbb{M}} f^X$

Each $X \in \mathbb{M}$ represents a unique forest of nested calls to $f$ — i.e. $f^{[X_1, \ldots, X_k]}$ corresponds to $k$ recursive calls to $f$ at top level, which make nested calls to $f$ with call-patterns $X_1, \ldots, X_k$ and so on.

# Computational Adequacy for $\mathrm{PCF}^R$

**Theorem** If $\mathcal{C}$ is a Lafont category with biproducts with $\mathcal{R} \subseteq \mathcal{R}_{\mathcal{C}}$, then for any program $M$

$$[\![M]\!] = \Sigma_{n \in \mathbb{N}}(W(M, \mathtt{n}) \cdot [\![\mathtt{n}]\!])$$

- Usual proof techniques (logical relations) depend on continuity.
- We use a translation into an equivalent operational semantics which nondeterministically assigns an explicit multiplicity to each variable — either a natural number or a nested multiset (for recursively defined variables). — cf. the resource $\lambda$-calculus.
- We define the denotational semantics of this system using our semantic approximants to the fixed point.

# Qualitative to Quantitative Models: Change of Base

Suppose $F : \mathcal{V} \to \mathcal{U}$ is a *monoidal functor*, with $m_{A,B} : FA \otimes FB \to F(A \otimes B)$ and $m_I : I \to FI$. Then for any $\mathcal{V}$-enriched category $\mathcal{C}$, $F$ induces a $\mathcal{U}$-enriched category $F\mathcal{C}$ by *change of base*:

- $Obj_{F\mathcal{C}} = Obj_{\mathcal{C}}$
- $F\mathcal{C}(A, B) = F(\mathcal{C}(A, B))$
- The identity and composition morphisms are $m_I; (F\mathrm{id}_A) : I \to F\mathcal{C}(A, B)$ and $m_{\mathcal{C}(A,B),\mathcal{C}(B,C)}; comp_{A,B,C} : F\mathcal{C}(A, B) \otimes F\mathcal{C}(B, C) \to \mathcal{C}(A, C)$.

Example — for any $\mathcal{V}$, the functor $\mathcal{V}(I, \_) : \mathcal{V} \to Set$ is monoidal — for any $\mathcal{V}$-category $\mathcal{C}$, change of base along this functor gives the *underlying category* $\mathcal{C}_0$.
Observe that $F$ acts as a functor from $\mathcal{C}_0$ to $(F\mathcal{C})_0$.

# From Coherence Spaces to $R^{\Pi}$

Consider the (strict monoidal) "forgetful" functor $\Phi_R$ from the category of coherence spaces and stable, linear maps (cliques of $A \multimap B$) to $R^{\Pi}$:

- $\Phi_R(|A|, \frown) = |A|$
- For $f : A \to B$, $\Phi_R(f) : |A| \times |B| \to R$ is the characteristic function of $R$ — $\Phi_R(f)(a, b) = 1$ if $(a, b) \in \mathcal{R}$, 0 otherwise.

Note that stability is crucial for functoriality.

# Coherence Enriched Categories

Many categories (hypercoherences, event structures, sequential algorithms...) bear a natural enrichment over coherence spaces. For example, games and history-free strategies (used by Abramsky and McCusker to interpret *Idealized Algol*):

- Objects are games.
- $\mathcal{G}(A, B)$ is the set $L_{A \multimap B}$ of complete (legal) sequences on $A \multimap B$, with $s \frown t$ if $s \sqcap t$ is even-length.
- $comp_{A,B,C}$ is the set of $((r, s), t) \in L_{A \multimap B} \times L_{B \multimap C} \times L_{B \multimap C}$ such that there exists $u$ with $u{\restriction}A \multimap B = r$, $u{\restriction}B \multimap C = s$ and $u{\restriction}A \multimap C = t$,

Stability of composition is the *Zipping Lemma*.

# $R$-weighted strategies

The underlying category $\Phi_R\mathcal{G}$ has games as objects, maps from $A$ to $B$ are maps from $L_{A\multimap B}$ to $R$.

- If $\mathcal{R}$ is the Boolean semiring $(\{\top,\bot\},\bigvee,\top,\wedge)$ then $\Phi_R\mathcal{G}$ is the category of games and nondeterministic strategies
- If $\mathcal{R}$ is the probability semiring $(\mathcal{R}_+^\infty,\Sigma,1,\times)$ then $\Phi_R\mathcal{G}$ is the category of probabilistic games and pre-strategies (Danos and Harmer).
- If $\mathcal{R}$ is the tropical semiring $(\mathbb{N}^\infty,\bigvee,0,+)$ then $\Phi_R\mathcal{G}$ is the category of slot games (Ghica).

# $\mathcal{R}$-weighted Idealized Algol

- Change of base induces/preserves categorical structure on games which we have used to define a categorical model of Idealized Algol (in particular, a *monoidal action* $\oslash : \mathcal{G}_S \times \mathcal{G} \to \mathcal{G}_S$ of $\mathcal{G}$ on a subcategory $\mathcal{G}_S$, such that the free ! is a *final coalgebra* for $J(A \oslash \_) \times I$).
- So we get a sound model of $IA^R$, with meaning-preserving functor from the qualitative model : computational adequacy proved as for $\mathrm{PCF}^R$ .
- *Full abstraction* follows easily from *definability of the basis* in Idealized Algol, proved by Abramsky and McCusker.