Probabilistic Coherence Spaces: the free exponential modality $!_f$

Michele Pagani*
joint work with:
Crubillé, Ehrhard, Tasson

* Institut de Recherche en Informatique Fondamentale
Université Paris Diderot (FR)
pagani@pps.univ-paris-diderot.fr

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How do we interpret $!A$?

- Every notion of ! associates an object A with a commutative comonoid:

$$\mathbf{1} \xleftarrow{w_{!A}} !A \xrightarrow{c_{!A}} !A \otimes !A$$

- Free commutative comonoid over A:
  - the “minimal” extension of A for getting a commutative comonoid
  - characterized by a universal property:

for every $(C, \mathrm{w}_C, \mathrm{c}_C)$ and $f : C \to A$,

$$\begin{array}{ccc} !_f A & \xrightarrow{\mathrm{der}_A} & A \\ \exists! f^\dagger \uparrow & \nearrow_{f} & \\ C & & \end{array}$$

Theorem (Lafont, 1988)

A $\star$-autonomous category $(\mathbb{L}, \otimes, \mathbf{1}, \bot)$ is a model of linear logic whenever:

1. *it has finite products and,*
2. *for every object A, there exists the free commutative comonoid $!_f A$ generated by A.*

Probabilistic Coherence Spaces (PCoh) (Danos, Ehrhard 2011)

Objects: $(|\mathcal{A}|, \mathrm{P}(\mathcal{A}))$, with $\mathrm{P}(\mathcal{A}) \subseteq (\mathbb{R}^+)^{|\mathcal{A}|}$ such that

- bipolar: $\mathrm{P}(\mathcal{A})^{\perp\perp} = \mathrm{P}(\mathcal{A})$,
  where $P^\perp = \{v \in (\mathbb{R}^+)^{|\mathcal{A}|} \; ; \; \forall u \in P, \langle v, u \rangle \leq 1\}$, for $P \subseteq (\mathbb{R}^+)^{|\mathcal{A}|}$
- complete: $\forall a \in |\mathcal{A}|, \exists v \in \mathrm{P}(\mathcal{A}), v_a \neq 0$
- bounded: $\forall a \in |\mathcal{A}|, \exists p \in \mathbb{R}^+, \forall v \in \mathrm{P}(\mathcal{A}), v_a \leq p$

Morphisms: matrices $\phi \in \mathbb{R}^{+|\mathcal{A}| \times |\mathcal{B}|}$ such that

- $\forall x \in \mathrm{P}(\mathcal{A}), \phi(x) \in \mathrm{P}(\mathcal{B})$

"Analytic" exponential modality

- web: $|!_a\mathcal{A}| = \mathcal{M}_f(|\mathcal{A}|)$
- cliques: $\mathrm{P}(!_a\mathcal{A}) = \{x^! \mid x \in \mathrm{P}(\mathcal{A})\}^{\perp\perp}$ (with $x^!_{[a_1,\dots,a_n]} = \prod_i x_{a_i}$)
- morphism: given $\phi \in \mathrm{PCoh}(\mathcal{A}, \mathcal{B})$, $(!_a\phi)_{\mu,[b_1,\dots,b_n]} = \sum_{\substack{(a_1,\dots,a_n), s.t. \\ [a_1,\dots,a_n]=\mu}} \prod_{i=1}^{n} \phi_{a_i,b_i}$

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titative semantics for higher-order probabilistic computation

Thomas Ehrhard* Michele Pagani+ Christine Tasson*

+Laboratoire d'Informatique de Paris Nord
Institut Galilée, Universitè de Paris Nord — Paris 13 (F
michele.pagani@lipn.univ-paris13.fr

*Laboratoire Preuves Programmes et Systèmes
Universit¬Diderot — — — Paris7(FR)
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Semantics of proofs and certified mathematics, IHP 2014

$(\mathbb{R}^+)^{|\mathcal{A}|}$

PCoh with the "analytic" exponentials is a Lafont category

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Semantics of proofs and certifi

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Ummm...
how do you prove that $!_a\mathcal{A}$ is the free commutative comonoid over $\mathcal{A}$?

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- morphism: given $\phi \in \mathrm{PCoh}(\mathcal{A}, \mathcal{B})$, $(!_a\phi)_{\mu,[b_1,\dots,b_n]} = \sum_{\substack{(a_1,\dots,a_n), s.t.\\ [a_1,\dots,a_n]=\mu}} \prod_{i=1} \phi_{a_i,b}$

Why supposing PCoh with $!_a$ a Lafont category ?

PCoh

- objects: $(|\mathcal{A}|, \mathrm{P}(\mathcal{A}))$ s.t.:
  $\mathrm{P}(\mathcal{A})$ bounded, complete $\subseteq \mathbb{R}^{+|\mathcal{A}|}$
  $\mathrm{P}(\mathcal{A})^{\perp\perp} = \mathrm{P}(\mathcal{A})$
- morphisms: $\phi \in \mathbb{R}^{+|\mathcal{A}|\times|\mathcal{B}|}$ s.t.:
  $\mathrm{P}(\mathcal{A}) \overset{\phi}{\mapsto} \mathrm{P}(\mathcal{B})$
- exponential modality:

$$!_a\mathcal{A} = (\mathcal{M}_{\mathrm{f}}(|\mathcal{A}|), \{x^! \mid x \in \mathrm{P}(\mathcal{A})\}^{\perp\perp}), \qquad (!_a\phi)_{\mu,[b_1,\ldots,b_n]} = \sum_{\substack{(a_1,\ldots,a_n)\\ [a_1,\ldots,a_n]=\mu}} \prod_{i=1}^{n} \phi_{a_i,b_i}$$

⇑ double glueing with focus = [0,1]

$\overline{\mathbb{R}}^{+}$-weighted Rel

- objects: sets $A, B, C, \ldots$
- morphisms: $\phi \in \overline{\mathbb{R}}^{+A\times B}$
- exponential modality: $!A = \mathcal{M}_{\mathrm{f}}(A)$,

$$(!\phi)_{\mu,[b_1,\ldots,b_n]} = \sum_{\substack{(a_1,\ldots,a_n)\\ [a_1,\ldots,a_n]=\mu}} \prod_{i=1}^{n} \phi_{a_i,b_i}$$

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- $\overline{\mathbb{R}}^{+}$-weighted Rel is a Lafont category, indeed:

$$!A = \bigoplus_{n=0}^{\infty} A^{\odot n}$$

- exponential modality:

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Wait! this reasoning cannot hold for PCoh:

- bi-products are not preserved under double glueing
- $\mathrm{P}\left(\mathrm{Bool}^{\odot 2}\right) \not\subseteq \mathrm{P}(!_a\mathrm{Bool})$,

  in fact, $e_{\mathtt{t}} \odot e_{\mathtt{f}} \in \mathrm{P}\left(\mathrm{Bool}^{\odot 2}\right)$

  but $(e_{\mathtt{t}} \odot e_{\mathtt{f}}) \oplus \mathbf{0} \notin \mathrm{P}(!_a\mathrm{Bool})$

lueing

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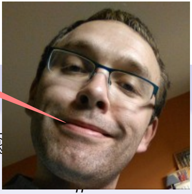

$\overline{\mathbb{R}}^{+}$-weighted Rel

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- morphisms: $\phi \in \overline{\mathbb{R}}$
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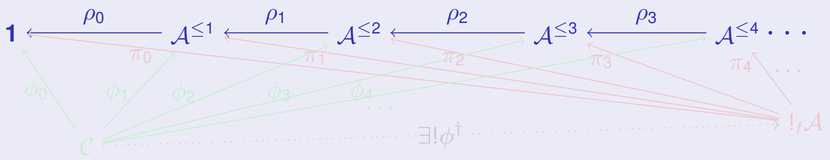
The free commutative comonoid $!_f\mathcal{A}$ of $\mathcal{A}$ is the limit of the following diagram:



- provided this limit exists and it commutes with the tensor product,
- and where $\mathcal{A}^{\leq n} = (\mathcal{A} \,\&\, \mathbf{1})^{\odot n}$.

We can compute:

- $|\mathcal{A}^{\leq n}| = \mathcal{M}_{\leq n}(|\mathcal{A}|), \quad \mathrm{P}\left(\mathcal{A}^{\leq n}\right) = \{\langle u_1, \ldots, u_n\rangle \mid \forall i \leq n, u_i \in \mathrm{P}\left(\mathcal{A}\right)\}^{\perp\perp}$

  where $\langle u_1, \ldots, u_n\rangle_{[a_1,\ldots,a_k]} = \dfrac{1}{n!}\displaystyle\sum_{\sigma\in S_n}\prod_{i=1}^{k}(u_{\sigma(i)})_{a_i} = \dfrac{(n-k)!}{n!}\sum_{f:k\hookrightarrow n}\prod_{i=1}^{k}(u_{f(i)})_{a_i}$

- $(\rho_n)_{\mu,\nu} = \begin{cases} 1 & \text{if } \mu = \nu \text{ and } \#\mu \leq n \\ 0 & \text{otherwise} \end{cases}$

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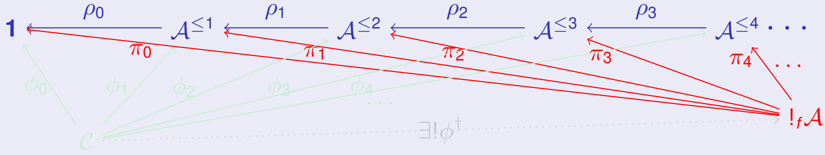

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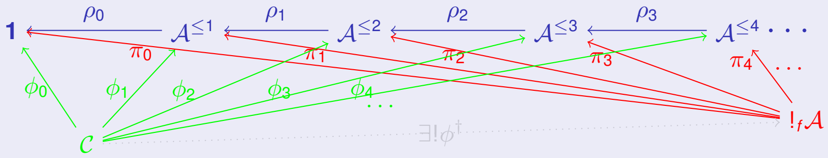

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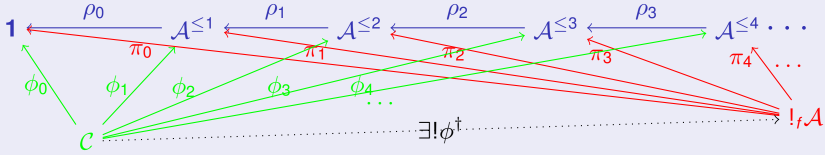

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Melliés, Tasson, Tabareau's receipt

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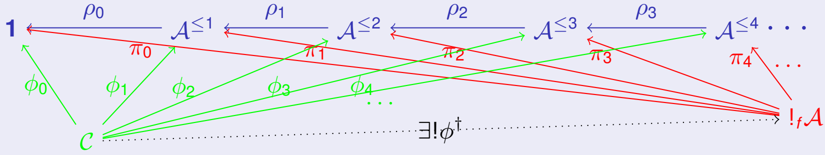

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$$\forall x \in \mathrm{P}(\mathcal{A}^{\leq n}), \forall k \leq n,\ x_{|_k} \in \mathrm{P}(\mathcal{A}^{\leq k})$$

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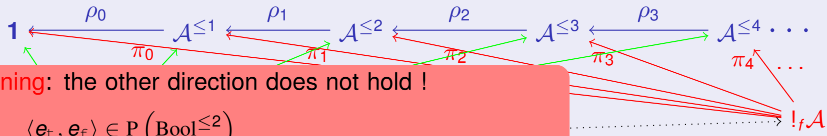

Warning: the other direction does not hold !

$$\langle e_t, e_f\rangle \in \mathrm{P}\left(\mathrm{Bool}^{\leq 2}\right)$$

but $\langle e_t, e_f\rangle \oplus \mathbf{0} \notin \mathrm{P}\left(\mathrm{Bool}^{\leq 3}\right)$,

$$\langle e_t, e_f\rangle \oplus \mathbf{0} \notin \mathrm{P}\left(\mathrm{Bool}^{\leq 4}\right),$$

$$\langle e_t, e_f\rangle \oplus \mathbf{0} \notin \mathrm{P}\left(\mathrm{Bool}^{\leq 5}\right),$$

$$\vdots$$

sor product,

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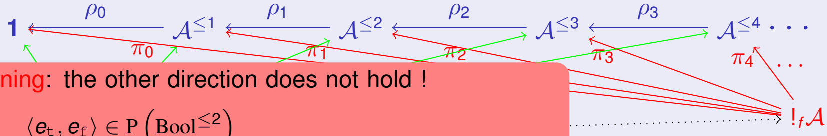

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Looking for the right coefficient $\mathrm{P}\left(\mathcal{A}^{\leq n}\right) \longrightarrow \mathrm{P}\left(!_f\mathcal{A}\right)$

- first try: $\iota_n \quad \begin{array}{ccc} \mathrm{P}\left(\mathcal{A}^{\leq n}\right) & & \mathrm{P}\left(\mathcal{A}^{\leq n+1}\right) \\ \langle u_1, \dots, u_n \rangle & \mapsto & \langle u_1, \dots, u_n, \mathbf{0} \rangle \end{array}$

$$(\iota_n)_{\mu,\nu} = \begin{cases} 1 - \frac{k}{n+1} & \text{if } \mu = \nu, \#\mu = k \leq n, \\ 0 & \text{otherwise.} \end{cases} \qquad \iota_{n,N} = \iota_n; \iota_{n+1}; \dots; \iota_{n+N}$$

  - $\lim_{N\to\infty} \prod_{n'=n+1}^{N} \left(1 - \frac{k}{n'}\right)$ diverges to $\mathbf{0}$
  - in fact, $\iota_{n,\infty}$: $\begin{array}{ccc} \mathrm{P}\left(\mathcal{A}^{\leq n}\right) & & \mathrm{P}\left(!_f\mathcal{A}\right) \\ \langle u_1, \dots, u_n \rangle & \mapsto & \langle u_1, \dots, u_n, \mathbf{0}, \mathbf{0}, \dots \rangle \sim \mathbf{0} \end{array}$

- second try: $\iota_{n,N} \quad \begin{array}{ccc} \mathrm{P}\left(\mathcal{A}^{\leq n}\right) & & \mathrm{P}\left(\mathcal{A}^{\leq N}\right) \\ \langle u_1, \dots, u_n \rangle & \mapsto & \langle u_1^q, \dots, u_n^q, \mathbf{0}^r \rangle \end{array}$, with $N = nq + r$ and $r < n$

$$(\iota_{n,N})_{\mu,\nu} = \begin{cases} \frac{(N-k)!q^k n!}{N!(n-k)!} & \text{if } \mu = \nu, \#\mu = k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

  - $\lim_{N\to\infty} \frac{(N-k)!q^k n!}{N!(n-k)!} = \frac{n!}{n^k(n-k)!}$
  - define:

$$\mathrm{P}\left(!_f\mathcal{A}\right) = \left\{ \lim_{N\to\infty} \iota_{n,N}(\langle u_1, \dots, u_n \rangle) \;\middle|\; n \in \mathbb{N}, \langle u_1, \dots, u_n \rangle \in \mathrm{P}\left(\mathcal{A}^{\leq n}\right) \right\}^{\perp\perp}$$

and check Melliés, Tasson, Tabareau's receipt.

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- second try: $\iota_{n,N} \quad \begin{array}{ccc} \mathrm{P}\left(\mathcal{A}^{\leq n}\right) & & \mathrm{P}\left(\mathcal{A}^{\leq N}\right) \\ \langle u_1, \ldots, u_n \rangle & \mapsto & \langle u_1^q, \ldots, u_n^q, \mathbf{0}^r \rangle \end{array}$, with $N = nq + r$ and $r < n$

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and check Melliés, Tasson, Tabareau's receipt.

Looking for the right coefficient $\mathrm{P}\left(\mathcal{A}^{\leq n}\right) \longrightarrow \mathrm{P}\left(!_f\mathcal{A}\right)$

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The end of the story

The “free” exponential modality

$$|!_f\mathcal{A}| = \mathcal{M}_{\mathrm{f}}(|\mathcal{A}|)\,, \qquad \mathrm{P}(!_f\mathcal{A}) = \{\langle\!\langle u_1, \ldots, u_n \rangle\!\rangle \mid u_i \in \mathrm{P}(\mathcal{A})\}^{\perp\perp}\,,$$

with $\langle\!\langle u_1, \ldots, u_n \rangle\!\rangle_{[a_1,\ldots,a_k]} = \frac{1}{n^k} \sum_{f:k \hookrightarrow n} \prod_{i=1}^{k} (u_{f(i)})_{a_i}$.

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The "analytic" exponential modality Danos&Ehrhard2011

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The "analytic" exponential modality — Danos&Ehrhard2011

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Theorem

The "free" and "analytic" exponential modalities are the same, i.e. $\mathrm{P}(!_f\mathcal{A}) = \mathrm{P}(!_a\mathcal{A})$.

Proof.

- $\mathrm{P}(!_f\mathcal{A}) \subseteq \mathrm{P}(!_a\mathcal{A})$, because $\langle\!\langle u_1, \ldots, u_n \rangle\!\rangle \leq (\sum_{i=1}^{n} \frac{1}{n} u_i)^!$
- $\mathrm{P}(!_a\mathcal{A}) \subseteq \mathrm{P}(!_f\mathcal{A})$, because $u^! = \lim_n \langle\!\langle \underbrace{u, \ldots, u}_{n} \rangle\!\rangle$

□