Graded Monads and Semantics of Effect Systems

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QSLC, Marseille, 1 Sep, 2016

Effect System

... is a framework to statically estimate side-effects:

$$\Gamma \vdash M : \tau \ \& \ e$$

1. Access analysis [Lucassen&Gifford '88, etc.]

$$\Gamma \vdash M : \tau \ \& \ \{rd(\rho), wr(\rho')\}$$

2. Communication analysis [Nielson&Nielson '96]

$$\Gamma \vdash M : \tau \ \& \ r!int \ . \ 0 + s?bool \ . \ r!int \ . \ 0$$

Wadler integrated effects and monadic types:

$$\left.\begin{array}{l}\Gamma \vdash M : \tau \ \& \ e \\ \Gamma \vdash M : T\tau\end{array}\right\} \Rightarrow \Gamma \vdash M : Te\tau$$

The Aim of this Research

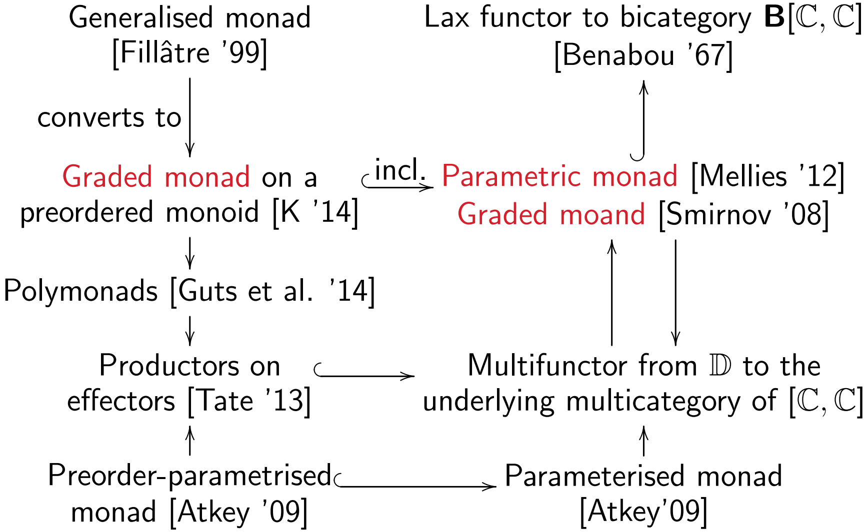
A question posed in [Wadler '98]:

What is the denotational semantics of $Te\tau$ in general?

We tackle this question.

- A general effect system **EFe**/**EFi** and their categorical semantics using graded monads
- A construction of graded monads by effect observation
- Resolutions of graded monads

Parameterisations of Monads



Generalising Effects

In many papers,

- Effects are ordered: to compare the extent / scope of effects.
- Effects are composable: to give the effect of the sequential execution.

$$M : Te\tau,\ N : Te'\sigma \Rightarrow \mathbf{let}\ x\ \mathbf{be}\ M\ \mathbf{in}\ N : T(e \cdot e')\sigma$$

The postulate on effects in this research

Effects form a preordered monoid.

$$\mathbb{E} = (E, \quad \lesssim, \quad 1 \in E, \quad (\cdot) : (E, \lesssim)^2 \to (E, \lesssim))$$

Effect System **EFe**

A calculus **EFe** for $\mathbb{E} = (E, \lesssim, 1, \cdot)$ consists of:

- Type

$$\tau ::= b \mid \tau \Rightarrow \tau \mid Te\tau \quad (b \in B, e \in E)$$

- Explicit subeffecting rule

$$\frac{\Gamma \vdash M : Te\tau \quad e \lesssim e'}{\Gamma \vdash T(e \lesssim e', M) : Te'\tau}$$

- Pure computation and sequential execution

$$\frac{\Gamma \vdash M : \tau}{\Gamma \vdash [M] : T1\tau} \qquad \frac{\Gamma \vdash M : Te\tau \quad \Gamma, x : \tau \vdash N : Te'\sigma}{\Gamma \vdash \mathbf{let}^{e,e'}\ x\ \mathbf{be}\ M\ \mathbf{in}\ N : T(e \cdot e')\sigma}$$

- Algebraic operation [c.f. Plotkin&Power] (omitted)

Graded Monad [Smirnov '08]

A graded monad for $\mathbb{E} = (E, \lesssim, 1, \cdot)$ on a category $\mathbb{C}$ consists of:

$$\begin{array}{rcrcll} T & : & (E, \lesssim) & \to & [\mathbb{C}, \mathbb{C}] & \\ T_{1,A} & : & A & \to & T1A & \text{(nat. on } A) \\ T_{e,e',A} & : & Te(Te'A) & \to & T(e \cdot e')A & \text{(nat. on } e, e', A) \end{array}$$

Graded Monad [Smirnov '08]

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The generalised monad laws:

$$\begin{array}{ccc} TeA & \longrightarrow & Te(T1A) \\ \downarrow & \diagdown\!\!\diagdown & \downarrow \\ T1(TeA) & \longrightarrow & TeA \end{array} \qquad \begin{array}{ccc} Te(Te'(Te''A)) & \longrightarrow & Te(T(e' \cdot e'')A) \\ \downarrow & & \downarrow \\ T(e \cdot e')(Te''A) & \longrightarrow & T(e \cdot e' \cdot e'')A \end{array}$$

(This is different from making Te a monad for each $e \in E$.)

Graded Monad [Smirnov '08]

Parametric effect monad $\to_\alpha$ Graded monad

A graded ring R comes with an $\mathbb{N}$-indexed family R_i of abelian groups such that

$$R_i R_j \subseteq R_{i+j}, \quad R = \bigoplus R_i$$

Such a family R_i forms a lax monoidal functor

$$R : (\mathbb{N}, 0, +) \to (\mathbf{Ab}, \mathbf{I}, \otimes)$$

Graded monad is a lax monoidal functor

$$T : \mathbb{E} \to ([\mathbb{C}, \mathbb{C}], \mathrm{Id}, \circ)$$

Graded Writer Monad

Consider the preordered monoid of languages over Σ:

$$\mathbb{E} = (P(\Sigma^*), \subseteq, \{\epsilon\}, \star) \quad (\star\text{: language concat.})$$

Graded Writer Monad:

$$\begin{array}{rcl} T & : & (P(\Sigma^*), \subseteq) \to [\mathbf{Set}, \mathbf{Set}] \\ TeA & = & e \times A \end{array}$$

$$\begin{array}{rcl} T_{1,A} & : & A \to T\{\epsilon\}A \\ T_{1,A} & = & \lambda a \,.\, (\epsilon, a) \end{array}$$

$$\begin{array}{rcl} T_{e,e',A} & : & Te(Te'A) \to T(e \star e')A \\ T_{e,e',A} & = & \lambda(s, (s', a)) \,.\, (ss', a) \end{array}$$

Graded State and Continuation Monads

$$\begin{array}{ll} \mathbb{E} = (E, \lesssim, 1, \cdot) & \text{preordered monoid} \\ S : (E, \lesssim) \to \mathbf{Set} & \text{any functor} \end{array}$$

The following end is a graded monad for $\mathbb{E}$ on **Set**:

$$SeA = \int_{d \in (E, \lesssim)} Sd \Rightarrow (A \times S(d \cdot e))$$

$$CeA = \int_{d \in (E, \lesssim)} (A \Rightarrow Sd) \Rightarrow S(e \cdot d)$$

Graded State and Continuation Monads

$$\begin{array}{ll} \mathbb{E} = (E, \lesssim, 1, \cdot) & \text{preordered monoid} \\ S : (E, \lesssim) \to \mathbf{Set} & \text{any functor} \end{array}$$

Theorem

For any parametrised monad $T : \mathbb{E}^{op} \times \mathbb{E} \times \mathbf{Set} \to \mathbf{Set}$, the following are graded monads for $\mathbb{E}$.

$$TeA = \int_d T(d, d \cdot e, A),$$

Graded Monads for Join Semilattices

Let $(E, \leq)$ be a join semilattice. The following are isomorphic data:

- A graded monad for $(E, \leq, \bot, \vee)$
- A functor of type $(E, \leq) \to \mathbf{Monad}(\mathbb{C})$

C.f. generalised monad [Fillâtre '99]

A Construction of graded monad

Can we construct graded monads from well-known structures?

A construction of graded monads on **Set** from effect observations.

Another aspects of effects

- Effects form an ordered algebra.
  - Access analysis: join semilattice with constants $rd(\rho), wr(\rho)$
- Effects are abstractions of side-effects.

We formulate these aspects of effects.

Effect Observation

... consists of the following data:

$$\alpha : T \longrightarrow (S, \sqsubseteq)$$

- $(S, \sqsubseteq)$ is a preordered monad over **Set** [K&Sato '13]
  — modeling an ordered algebra of effects
- T is a **Set**-monad
  — modeling side-effects of a language
- $\alpha : T \to S$ is a monad morphism
  — modeling the abstraction of side-effects

Graded Monads from Effect Observations

From

$$\alpha : T \longrightarrow (S, \sqsubseteq)$$

We construct a monoid $\mathbb{E}$:

$$\mathbb{E} = (S1, \sqsubseteq_1, 1, \star)$$

$$1 = 1 \xrightarrow{\eta_1} S1 \quad e \star e' = 1 \xrightarrow{e} S1 \xrightarrow{(e')^\#} S1$$

and a graded monad D:

$$DeA = \{c \in TA \mid \alpha_1 \circ T!_A(c) \sqsubseteq_1 e\}$$

$$TA \xrightarrow{T!_A} T1 \xrightarrow{\alpha_1} S1$$

Graded Writer Monad

From

$$\{-\} : Wr \longrightarrow (P \circ Wr, \subseteq)$$

We construct a monoid $\mathbb{E}$:

$$\mathbb{E} = (P(Wr1), \subseteq, 1, \star) \simeq (P(\Sigma^*), \subseteq, \{\epsilon\}, \star)$$

and a graded monad D:

$$DeA = \{(w, c) \in \Sigma^* \times A \mid \{(w, *)\} \in e\} \simeq e \times A$$

Graded Monad for Effect Analysis

Let Ω be a ranked alphabet.

$$\mathbb{E} = (P(|\Omega| + 1), \cdots)$$

Meaning of effects:

- $\{f, g, *\}$: May perform f, g, or return a value
- $\{f, g, c\}$: May perform f, g, or c (but no return value)

$$DeA = \{c \in T_\Omega A \mid ops(c) \subseteq e\}$$

$$ops(c) = \{o \mid o \text{ occurs in } c\} \cup \{* \mid \text{c is not closed}\}$$

Graded Writer Monad

From

$$|-|: T_\Omega \longrightarrow (P(|\Omega| + -), \subseteq)$$

$$|x| = \{x\}, \quad |o(t_1, \cdots, t_n)| = \{o\} \cup |t_1| \cup \cdots \cup |t_n|$$

We construct a monoid $\mathbb{E}$:

$$\mathbb{E} = (P(|\Omega| + 1), \subseteq, 1, \star)$$

whose multiplication is

$$\{f, g, *\} \star \{f, p, q, *\} = \{f, g, p, q, *\}$$

and a graded monad D:

$$D e I = \{c \in T_\Omega I \mid |c[*/i]_{i \in I}| \subseteq e\}$$

Graded Writer Monad

From

$$|-|: T_\Omega \longrightarrow (P(|\Omega| + -), \subseteq)$$

$$|x| = \{x\}, \quad |o(t_1, \cdots, t_n)| = \{o\} \cup |t_1| \cup \cdots \cup |t_n|$$

We construct a monoid $\mathbb{E}$:

$$\mathbb{E} = (P(|\Omega| + 1), \subseteq, 1, \star)$$

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and a graded monad D:

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Effect System **EFi**

A calculus **EFi** for $\mathbb{E} = (E, \lesssim, 1, \cdot)$ consists of:

- Types are the same as **EFe**.
- Implicit subeffecting rule

$$\frac{\Gamma \vdash M : Te\tau \quad e \lesssim e'}{\Gamma \vdash M : Te'\tau}$$

- Pure computation and sequential execution

$$\frac{\Gamma \vdash M : \tau}{\Gamma \vdash [M] : T1\tau} \qquad \frac{\Gamma \vdash M : Te\tau \quad \Gamma, x : \tau \vdash N : Te'\sigma}{\Gamma \vdash \textbf{let } x \textbf{ be } M \textbf{ in } N : T(e \cdot e')\sigma}$$

- Algebraic operation [c.f. Plotkin&Power] (omitted)

How do we interpret this language?

Effect Erasure Function

We capture the following situation categorically.

$$\text{The set of } \mathbf{EFi} \text{ types} \xrightarrow{|-|} \text{The set of } \lambda_{ML} \text{ types}$$

$$\begin{aligned} |b| &= b \\ |\tau \Rightarrow \tau'| &= |\tau| \Rightarrow |\tau'| \\ |Te\tau| &= T|\tau| \end{aligned}$$

Here, λ_{ML} is Moggi's computational metalanguage ($=\lambda^{\rightarrow}+$monadic types $T\tau$).

A view on EFi types

EFi types refine λ_{ML} types.

Categorical Semantics of **EFi**

... is given by the following structure:

$$\begin{array}{c} \text{The set of } \textbf{EFi} \text{ types} \\ \Big\downarrow |-| \\ \text{The set of } \lambda_{ML} \text{ types} \end{array}$$

$$\begin{aligned} |b| &= b \\ |\tau \Rightarrow \tau'| &= |\tau| \Rightarrow |\tau'| \\ |Te\tau| &= T|\tau| \end{aligned}$$

Categorical Semantics of **EFi**

... is given by the following structure:

$$\mathbb{P} \circlearrowleft \dot{T}$$

$$|{-}| \downarrow$$

The set of λ_{ML}

$$\begin{aligned} |b| &= b \\ |\tau \Rightarrow \tau'| &= |\tau| \Rightarrow |\tau'| \\ |Te\tau| &= T|\tau| \end{aligned}$$

$(\mathbb{P}, \dot{T})$ CCC with a strong graded monad

Categorical Semantics of **EFi**

... is given by the following structure:

$$
\begin{array}{c}
\mathbb{P} \circlearrowright {\color{red}\dot{T}} \\
{\scriptstyle |-|}\big\downarrow \\
\mathbb{C} \circlearrowright T
\end{array}
\qquad
\begin{aligned}
|b| &= b \\
|\tau \Rightarrow \tau'| &= |\tau| \Rightarrow |\tau'| \\
|Te\tau| &= T|\tau|
\end{aligned}
$$

$(\mathbb{P}, \dot{T})$ CCC with a strong graded monad
$(\mathbb{C}, T)$ CCC with a strong monad

Categorical Semantics of **EFi**

... is given by the following structure:

$$\begin{array}{c} \mathbb{P} \circlearrowright \dot{T} \\ p \downarrow \\ \mathbb{C} \circlearrowright T \end{array} \qquad \begin{aligned} |b| &= b \\ |\tau \Rightarrow \tau'| &= |\tau| \Rightarrow |\tau'| \\ |Te\tau| &= T|\tau| \end{aligned}$$

$(\mathbb{P}, \dot{T})$	CCC with a strong graded monad
$(\mathbb{C}, T)$	CCC with a strong monad
$p : \mathbb{P} \to \mathbb{C}$	Faithful, strictly preserving CC structure and mapping $\dot{T}$ to T ($p(\dot{T}eA) = T(pA)$ etc.)

Categorical Semantics of **EFi**

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$(\mathbb{P}, \dot{T})$	CCC with a strong graded monad
$(\mathbb{C}, T)$	CCC with a strong monad
$p : \mathbb{P} \to \mathbb{C}$	Faithful, strictly preserving CC structure and mapping $\dot{T}$ to T ($p(\dot{T}eA) = T(pA)$ etc.)

Theorem: Soundness of the semantics

$\Gamma \vdash M : \tau$ implies

$$\begin{array}{ccc} [\![\Gamma]\!] & \xrightarrow{\exists! m} & [\![\tau]\!] \\ & & \\ \langle|\Gamma|\rangle & \xrightarrow[\langle M \rangle]{} & \langle|\tau|\rangle \end{array} \qquad \begin{array}{c} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{array}$$

EFi Semantics by $\top\top$-Lifting

How to give a graded monad $\dot{T}$ on $\mathbb{P}$?

$$\begin{array}{c} \mathbb{P} \\ \downarrow p \\ T \circlearrowright \mathbb{C} \end{array}$$

$\mathbb{P}$: CCC
$\mathbb{C}$: CCC, T : strong monad
p : preordered fibration with fibred meets
strictly preserves CC structure

For any $R \in \mathbb{C}$ and $S : (E, \lesssim) \to \mathbb{P}_{TR}$, the inverse image yields a graded monad $\dot{T}$ which is mapped to T.

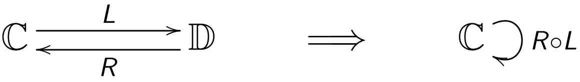
$$\dot{T}eX \dashrightarrow \bigwedge_{d \in E}(X \mathrel{\dot{\Rightarrow}} Sd) \mathrel{\dot{\Rightarrow}} S(e \cdot d)$$

$$T(pX) \xrightarrow[bind]{} (pX \Rightarrow TR) \Rightarrow TR$$

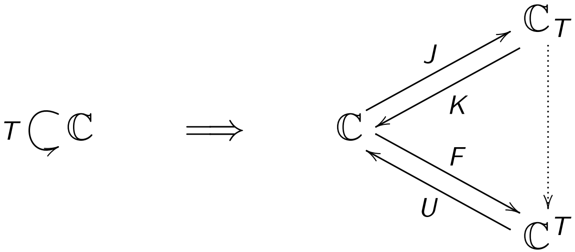
$$\begin{array}{c} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{array}$$

Resolutions of Monads

An adjunction yields a monad:



A monad yields two adjunctions:

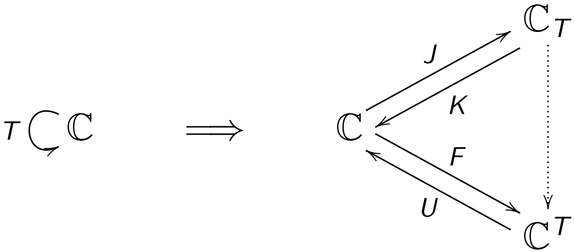


Resolutions of Monads

An adjunction transports a monad:

$$\mathbb{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{D} \circlearrowright T \quad \Longrightarrow \quad \mathbb{C} \circlearrowright R \circ T \circ L$$

A monad yields two adjunctions:



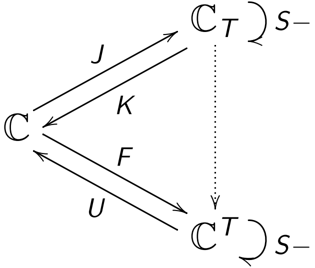
Resolutions of Graded Monads

An adjunction transports a graded monad:

$$\mathbb{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{D} \circlearrowright T- \Longrightarrow \qquad \mathbb{C} \circlearrowright R\circ(T-)\circ L(=)$$

A graded monad yields two adjunctions with twists:

$$T- \circlearrowleft \mathbb{C} \qquad \Longrightarrow$$



Eilenberg-Moore Like resolution

... of a graded monad T uses the category $\mathbb{C}^T$ of graded algebras of T:

$$A : (E, \lesssim) \to \mathbb{C}, \quad a_{e,e'} : Te(Ae') \to A(ee').$$

$$\begin{array}{ccc} Ae & \xrightarrow{\eta} & T1(Ae) \\ & \searrow^{=} & \downarrow a \\ & & Ae \end{array} \qquad \begin{array}{ccc} Te(Te'(Ae'')) & \xrightarrow{Tea} & Te(A(e'e'')) \\ \mu \downarrow & & \downarrow a \\ T(ee')(Ae'') & \xrightarrow[a]{} & A(ee'e'') \end{array}$$

It comes with a twist functor $S : (E, \lesssim) \to [\mathbb{C}^T, \mathbb{C}^T]$:

$$Sd(A, a) = (\lambda e\ .\ A(ed), \lambda e, e'\ .\ a_{e,e'd})$$

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It comes with a twist functor $S : (E, \lesssim) \to [\mathbb{C}^T, \mathbb{C}^T]$:

$$S1 = \mathrm{Id}, \quad S(dd') = Sd \circ Sd'$$

Kleisli-Like Resolution

... of a graded monad T uses the following category $\mathbb{C}_T$:

Object (e, A) where $e \in \mathbb{E}, A \in \mathbb{C}$

Morphism The homset $\mathbb{C}_T((e, A), (e', B))$ is

$$\int^{d \in \mathbb{E}} \mathbb{E}(ed, e') \times \mathbb{C}(A, TdB)$$

It comes with a twist functor $S : (E, \lesssim) \to [\mathbb{C}_T, \mathbb{C}_T]$

$$Sd(e, A) = (ed, A)$$

Kleisli-Like Resolution

... of a graded monad T uses the following category $\mathbb{C}_T$:

Object (e, A) where $e \in \mathbb{E}, A \in \mathbb{C}$

Morphism The homset $\mathbb{C}_T((e, A), (e', B))$ is

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It comes with a twist functor $S : (E, \lesssim) \to [\mathbb{C}_T, \mathbb{C}_T]$

$$S1 = \mathrm{Id}, \quad S(dd') = Sd \circ Sd'$$

Category of Resolution of Graded Monads

Let T be an $\mathbb{E}$-graded monad on $\mathbb{C}$.

Object consists of

$$\mathbb{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbb{D} \circlearrowright S-$$

such that $S1 = \mathrm{Id}$ and $S(dd') = Sd \circ Sd'$

Morphism Map of adjunctions and twists.

Theorem (Fujii, K, Melliès '16)

$\mathbb{C}_T$ and $\mathbb{C}^T$ are initial and final in the category of resolutions of T.

Action of a Monoidal Category

A monoid homomorphism

$$f : E \to (C \Rightarrow C, \mathrm{id}_C, \circ)$$

is nothing but a monoid action of E on C.
Its categorical analogue is a strong monoidal functor

$$F : \mathbb{E} \to ([\mathbb{C}, \mathbb{C}], \mathrm{Id}, \circ).$$

- Called actegory (McCrudden) / $\mathbb{E}$-category (Pareigis).
- Kelly and Janelidze studies when F has a right adjoint.

(Op)Lax Action of a Monoidal Category

For a lax monoidal $F : \mathbb{E} \to ([\mathbb{C}, \mathbb{C}], \mathrm{Id}, \circ)$,

- Durov employed it as a generalisation of graded ring.
- Smirnov studies it under the name graded monad.
- It is mentioned in [Atkey '08].
- A calculus similar to **EFe** / **EFi** appears in the APPSEM paper by Benton et al.
- Melliès studies it under the name parametric monad / negative E-category.

For an oplax monoidal $F : \mathbb{E} \to ([\mathbb{C}, \mathbb{C}], \mathrm{Id}, \circ)$,

- See recent work on coeffects [Petricek, Orchard, Mycroft '13], [Brunel, Gaboardi, Mazza, Zdancewic '14], [Ghica, Smith '14] and bounded LL.

Conclusion

- Categorical semantics of **EFe** / **EFi** using graded monads
- A construction of graded monad from effect observation
- A construction of graded monad for the semantics of **EFi**
- Eilenberg-Moore / Kleisli resolutions of graded monads

Thank you!