Infinitary proof theory : the multiplicative additive case

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Introduction

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- **Proofs-programs over these data types**

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- **Proofs-programs over these data types**

$$\begin{aligned} double(n) \quad &= \quad 0 && \text{if } n = 0 \\ &= \quad succ(succ(double(m))) && \text{if } n = succ(m) \end{aligned}$$

$$\Pi_{double} = \cfrac{\cfrac{\cfrac{\cfrac{\overline{1 \vdash 1}\ (1)}{1 \vdash 1 \oplus N}\ (\oplus_1)}{1 \vdash N}\ (\mu_l) \qquad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Pi_{double}}{N \vdash N}}{N \vdash 1 \oplus N}\ (\oplus_2)}{N \vdash N}\ (\mu_r)}{N \vdash 1 \oplus N}\ (\oplus_2)}{N \vdash N}\ (\mu_r)}{1 \oplus N \vdash N}\ (\oplus_l)}{N \vdash N}\ (\mu_l)$$

Infinitary proofs in the literature

- **Verification device**: Complete deduction sytem giving algorithms for checking validity (Tableaux, sequent calculi).
- **Completeness arguments**: Intermediate objects between syntax and semantics (Kozen, Kaivola, Walukiewicz).

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- **Completeness arguments**: Intermediate objects between syntax and semantics (Kozen, Kaivola, Walukiewicz).
- **But rarely seen as proof/program objects.**

Structural proof theory

Two main properties:

- Syntactic cut-elimination

- Focalization

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Infinitary proof system $\mu MALL^{\infty}$

Formulas

$\mu MALL^\infty$ formulas

$$\begin{array}{lll} F ::= & X \mid \top \mid \bot \mid 0 \mid 1 \mid F \otimes F \mid F ⅋ F \mid F \& F \mid F \oplus F & \textit{MALL} \\ \mid & \mu X.F & \textbf{least fixed point} \\ \mid & \nu X.F & \textbf{greatest fixed point} \end{array}$$

- μ and ν are dual.

  Example: $\neg(\nu X.X \otimes X) = \mu X.X ⅋ X$.

- Data types encoding

$$\begin{array}{rcl} \text{Nat} & := & \mu X.1 \oplus X \\ \text{Stream}(A) & := & \nu X.A \otimes X \end{array}$$

Sequent calculus

$\mu MALL^\infty$ pre-proofs are the trees coinductively generated by:

Usual logical rules

$$\frac{\vdash \Gamma, F \quad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G} (\otimes) \qquad \frac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G} (\parr) \qquad \frac{\vdash \Gamma, F \quad \vdash \Gamma, G}{\vdash \Gamma, F \& G} (\&) \qquad \frac{\vdash \Gamma, F_i}{\vdash \Gamma, F_1 \oplus F_2} (\oplus_i)$$

Identity rules

$$\frac{}{\vdash F, \neg F} (\mathtt{ax}) \qquad \frac{\vdash \Gamma, F \quad \vdash \Delta, \neg F}{\vdash \Gamma, \Delta} (\mathtt{cut})$$

Rules for μ and ν

$$\frac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F} (\mu) \qquad \frac{\vdash \Gamma, F[\nu X.F/X]}{\vdash \Gamma, \nu X.F} (\nu)$$

Sequent calculus

$$
\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\,(\mu)}{\vdash \mu X.X}\,(\mu) \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\,(\nu)}{\vdash \nu X.X, F}\,(\nu)}{\vdash F}\,(\mathtt{cut})
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Pre-proofs are unsound, hence the need for a validity condition.

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Validity condition

A **proof** is a pre-proof in which every infinite branch unfolds a ν formula infinitely often.

Focalization

Focalization in MALL

Classify the connectives into 2 categories:

- **Negative connectives:** Invertible connectives ie. we don't lose provability by applying these rules ($⅋, \&$).

  If $\vdash \Gamma, A ⅋ B$ is provable then $\vdash \Gamma, A, B$ is also provable.

- **Positive connectives:** Non Invertible connectives ie. there is a choice to make, a bad choice may lead to a loss of provability ($\oplus, \otimes$).

$$\frac{\vdash \bot}{\vdash \top \oplus \bot}\ (\oplus)_2 \qquad\qquad \frac{\vdash X \quad \vdash 1, X^\perp}{\vdash X \otimes 1, X^\perp}\ (\otimes)$$

Focalization in MALL

Complete strategy to prove a sequent Γ:

Negative phase	Positive phase
Γ contains a negative formula	**Γ contains no negative formula**
choose a negative formula and apply the unique negative rule available.	choose some positive formula and decompose it hereditarily until negative subformulas are reached.

Focused proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}\,(ax)}{\vdash B, D \oplus B^\perp}\,(\oplus) \qquad \dfrac{\dfrac{}{\vdash C, C^\perp}\,(ax)}{\vdash C, D \oplus C^\perp}\,(\oplus)}{\vdash B \otimes C, D \oplus B^\perp, D \oplus C^\perp}\,(\otimes)}{\vdash A \oplus (B \otimes C), D \oplus B^\perp, D \oplus C^\perp}\,(\oplus)}{\vdash A \oplus (B \otimes C), (D \oplus B^\perp) \parr (D \oplus C^\perp)}\,(\parr)$$

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Non-Focused proof:

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$$

Focalization for $\mu MALL^{\infty}$

Classification of connectives

ν is classified **negative** and μ were classified **positive**.

If μ is classified negative, we would have

$$\dfrac{\dfrac{\vdash \quad \vdots}{\vdash \top \otimes \top, \mu X.X}\ (\mu)}{\vdash \top \otimes \top, \mu X.X}\ (\mu)$$

... which is not a valid proof.

Proof of completeness of focalization for MALL

Transform a MALL proof into a focused proof by using:

- **Commutation of Negatives:** negative connectives commute under all other connectives.
- **Commutation of positives:** positive connectives commute under each other.

Theorem

There is a process based on these commutations that ***terminates*** *on a focused proof.*

Proof of completeness of Focalization for $\mu MALL^{\infty}$

Works in the same way, under some adaptations.

Theorem

*There is a process based on a commutation rules which is **productive**.*

Theorem

This process produces a focused proof.

Cut elimination

Cut elimination procedure - External operations

$$\dfrac{\dfrac{\vdash \Delta, F, G}{\vdash \Delta, F \parr G}\,(\parr) \qquad \dots}{\vdash \Sigma, F \parr G}\,(\mathtt{cut}) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F, G \qquad \dots}{\vdash \Sigma, F, G}\,(\mathtt{cut})}{\vdash \Sigma, F \parr G}\,(\parr)$$

$$\dfrac{\dfrac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F}\,(\mu) \qquad \dots}{\vdash \Sigma, \mu X.F}\,(cut) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F[\mu X.F/X] \qquad \dots}{\vdash \Sigma, F[\mu X.F/X]}\,(cut)}{\vdash \Sigma, \mu X.F}\,(\mu)$$

External operations are productive

Cut elimination procedure - Internal operations

$$\dfrac{\dfrac{\vdash \Delta, F_2 \qquad \vdash \Delta, F_1}{\vdash \Delta, F_2 \& F_1}\ (\&) \qquad \dfrac{\vdash \Gamma, F_i^\perp}{\vdash \Gamma, F_1^\perp \oplus F_2^\perp}\ (\oplus_i)}{\vdash \Sigma}\ (\mathtt{cut})$$

$$\Rightarrow \quad \dfrac{\vdash \Delta, F_i \qquad \vdash \Gamma, F_i^\perp}{\vdash \Sigma}\ (\mathtt{cut})$$

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$$\Rightarrow \quad \dfrac{\vdash \Delta, F[\mu X.F/X] \qquad \vdash \Gamma, F^\perp[\nu X.F^\perp/X]}{\vdash \Sigma}\ (\mathtt{cut})$$

Internal operations are not productive

Cut elimination algorithm

- Internal phase:
  Perform internal operations while you can't do anything else.
- External phase:
  Produce a part of the output tree whenever you can.

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Cut elimination is productive

Theorem

The internal phase always halts.

Cut elimination is productive

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Proof sketch: Suppose that the internal phase diverges for a proof π of $\vdash \Delta$.

- Let θ be the sub-derivation of π explored by the reduction.
- Extract from θ a proof of the empty sequent.
- We define a truth semantics for $\mu MALL^{\infty}$ formulas and show that the proof system is sound with respect to it. Contradiction.

Cut elimination produces a proof

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Follows the same proof idea.

Conclusion

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- **Contributions:**

  Foundations for multiplicative additive infinitary proof theory:
  Syntactic cut elimination and Focalization.

- **Future work:**
  - Go beyond Linear Logic and handle structural rules.
  - Translate infinitary proofs to finitary ones.
  - Same question while preserving the computational content.

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Thank you for your attention!