One-Dimensional Logic over Words

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One-dimensional logic

- Fragment of first-order logic, denoted F_1
- Quantification restricted to blocks of existential (universal) quantifiers that leave at most one variable free
  - $\exists y_1 \exists y_2 \ldots \exists y_k \; \varphi(x, y_1, y_2, \ldots, y_k)$
  - $\forall y_1 \forall y_2 \ldots \forall y_k \; \varphi(x, y_1, y_2, \ldots, y_k)$

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- 0-dimensional subformulas allowed:
  - $\forall xyz(Txy \wedge Tyz \rightarrow Txz)$
- Non-example:
  - $\forall xy(x < y \rightarrow \exists z(x < z \wedge z < y))$
- $\mathrm{FO}^2 \subseteq \mathrm{F}_1$

Satisfiability over arbitrary structures structures

- The satisfiability problem for F_1 over the class of all relational structures is undecidable.
  - Simple grid axiomatisation Φ_{grid}:
    - $\exists x Origo(x) \wedge \forall x \exists y Hxy \wedge \forall x \exists y Vxy$
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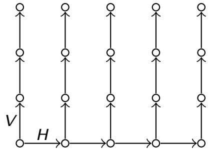
  - Every model of Φ_{grid} homomorphically embeds the standard grid:



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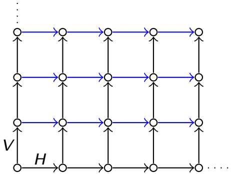

Regaining decidability

Two ways:

- Restricting the syntax
  - Uniform one-dimensional fragment UF_1:
    - boolean combinations of atoms $Sx_1, \ldots, x_k$ and $Ty_1, \ldots, y_l$ of arity greater than 1 only if $\{x_1, \ldots, x_k\} = \{y_1, \ldots, y_l\}$
    - equalities and unary atoms can be used freely
    - Still embeds FO^2
    - Finite model property, NExpTime-complete satisfiability problem
    - A series of papers Hella, Kuusisto (AIML'15); K., Kuusisto (MFCS 15); K., Kuusisto (CSL 15)
- Restricting the class of structures
  - words: this paper
  - trees: ongoing work

Words

One-dimensional logic over words, $\mathrm{F}_1[<, +1]$:

- Signature:
  - binary navigational relations: $<$, $+1$
  - possibly infinite set of unary symbols
- Words:
  - finite structures
  - $<$ is a linear order, $+1$ is the induced successor relation
  - multiple unary relations may hold at a single position
- ω-words
  - infinite structures
  - the order isomorphic to $(\mathbb{N}, <, +1)$

Two-variable logic over words

Two-variable logic over words and ω-words, $\mathrm{FO}^2[<, +1]$.
Etessami, Vardi, Wilke (Inf. & Comp. 2002):

- Expressively equivalent to unary temporal logic, UTL (temporal logic with operators *next*, *previously*, *sometime in the future*, *sometime in the past*)
- exponentially more succinct than UTL
- Exponential model property: every φ satisfiable by a word (ω-word) is satisfiable by a word of size exponential in $|\varphi|$ (ω-word $\mathfrak{M}_1\mathfrak{M}_2^\omega$ with $|\mathfrak{M}_i|$ bounded exponentially in $|\varphi|$).
- NExpTime-complete satisfiability problem

Main results

- $\mathrm{F}_1[<,+1]$ expressively equivalent to $\mathrm{FO}^2[<,+1]$
  - and thus also to UTL
  - (also to $\mathrm{C}^2[<,+1]$)
- Exponential model property
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- Exponential model property
- NExpTime-complete satisfiability problem (both over words and ω-words)
  - the same as $\mathrm{FO}^2[<, +1]$?
  - yes, but $\mathrm{F}_1[<, +1]$ allows for simpler (and shorter?) definitions of some properties

Example 1

There are three points satisfying together all of $P_1, \dots, P_n$:

$$\exists x_1, x_2, x_3 \bigwedge_{i=1}^{n} (P_i x_1 \vee P_i x_2 \vee P_i x_3).$$

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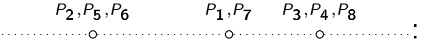
Easy to write an FO^2 (or FO^1) formula if you know how $P_1, \dots, P_n$ are divided among x_1, x_2, x_3:

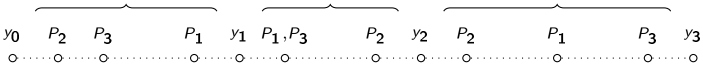


$$\exists x (P_2 x \wedge P_5 x \wedge P_6 x) \wedge \exists x (P_1 x \wedge P_7 x) \wedge \exists x (P_3 x \wedge P_4 x \wedge P_8 x)$$

But we need to consider all possible divisions (exponentially many)...

Example 2

There are m non-overlapping intervals in each of which each of P_i holds:



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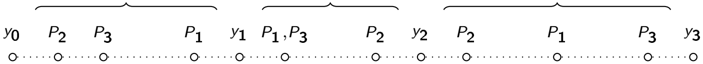

In $\mathrm{F}_1[<, +1]$:

$$\exists_{y_0 y_1 \dots y_n x_{11} \dots x_{1n} \dots x_{m1} \dots x_{mn}} \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} (y_{i-1} \leq x_{ij} \wedge x_{ij} < y_i \wedge P_j x)$$

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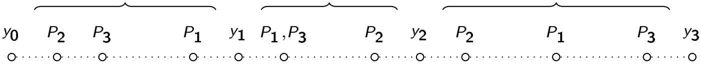

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In $\mathrm{FO}^2[<, +1]$, knowing the order in which P_is are satisfied:

$$\exists x (P_2 x \wedge \exists y (x < y \wedge P_3 y \wedge \exists x (y < x \wedge P_1 x \wedge \dots))))$$

But all possible orders must be considered...

Expressivity: proof idea

- Take a subformula $\psi \equiv \exists y_1 \dots y_k \psi_0(y_0, y_1, \dots, y_k)$
- Convert ψ_0 into DNF, distribute $\exists y_1 \dots y_k$ over $\vee$:

$$\psi \equiv \bigvee_{i=1}^{l} \exists y_1 \dots, y_k \psi_i(y_0, y_1, \dots, y_k)$$

- Take a single disjunt $\exists y_1 \dots y_k \psi_i$. Let π fully specifies the relations among $y_0, \dots, y_k$. Consider all possible π:

$$\psi_i \equiv \bigvee_{\pi} (\pi(y_0, \dots, y_k) \wedge \psi_i^{\pi}(y_0, y_1, \dots, y_k))$$

Expressivity: proof idea, cntd.

- Distribute again $\exists$ over $\vee$ and take a single disjunct:

$$\psi' = \exists y_1 \ldots, y_k(\pi(y_0, \ldots, y_k) \wedge \psi_i^\pi(y_0, y_1, \ldots, y_k))$$

- Arrange ψ_i^π into $\bigwedge_{j=0}^k \psi_{i,j}^\pi(y_j)$
  - $\psi_{i,j}^\pi(y_j)$ – conjuncts with free variable y_j
  - Assume that π implies that the variables appear in order $y_3 \; {\scriptstyle (+1)} \; y_2 \ll y_0 \ll y_1 \ll y_4$:
- $\psi' \equiv \psi'^\pi_{i,0}(x) \wedge$
  $\exists y(y \ll x \wedge \psi'^\pi_{i,2}(y) \wedge \exists x(+1(x, y) \wedge \psi'^\pi_{i,3}(x))) \wedge$
  $\exists y(x \ll y \wedge \psi'^\pi_{i,1}(y) \wedge \exists y_0(y \ll x \wedge \psi'^\pi_{i,4}(x)))$

Expressivity: summary

- We gave a translation from $\mathrm{F}_1[<, +1]$ to $\mathrm{FO}^2[<, +1]$ with an exponential blow-up
- Main sources of the blow-up:
  - transformation to DNF
  - considering all possible orderings of $y_0, \ldots, y_n$

Complexity: proof idea

- We show exponential model property
  - this leads to NExpTime-upper bound on the complexity of the satisfiability problem
  - as in the case of $\mathrm{FO}^2[<, +1]$
- Generalisation of Scott normal form for FO^2:
  conjunctions of
  - $\forall y_1, \ldots, y_n \psi(y_1, \ldots, y_n)$
  - $\forall x \exists y_1, \ldots, y_k \psi(x, y_1, \ldots, y_k)$

  where ψ is quantifier-free
- Classical contraction technique:
  - Take $\mathfrak{M} \models \varphi$, for a normal form φ
  - Find two *similar* positions w_1, w_2 in $\mathfrak{M}$
  - Remove the fragment of the word between w_1 and w_2

Complexity: proof idea - cntd.

Raughly:

- For a given normal form φ, $\mathfrak{M} \models \varphi$ two positions $w_1, w_2 \in M$ are *similar* if:
  - They agree on unary predicates,
  - The elements $w_1 + i$ and $w_2 + i$ agree on unary predicates for $i \in \{-m, \ldots, m\}$, where m is the width of φ
  - For $i \in \{1, \ldots, m\}$, $w_1 + i$ and $w_2 + i$ can see the same *patterns* of elements to the right
  - For $i \in \{-m, \ldots, -1\}$, $w_1 + i$ and $w_2 + i$ can see the same *patterns* of elements to the left

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- If a model $\mathfrak{M} \models \varphi$ is longer than $\mathfrak{f}(|\varphi|)$ for some fixed exponential $\mathfrak{f}$ then it must have two similar nodes.
- This allows to shorten finite models to size exponential in $|\varphi|$
- An appropriate surgery is possible also for ω-words

Limits of decidability

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  - $\mathrm{F}_1[+1,B]$ and $\mathrm{F}_1[<,B]$ undecidable
- Two linear orders accessible by the successor relations
  - $\mathrm{F}_1[+1_a,+1_b]$ undecidable

Conclusions and future work

Summary:

- Over words and ω-words $\mathrm{F}_1[<,+1] \equiv \mathrm{FO}^2[<,+1]$
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Future work:

- Succinctness of $\mathrm{F}_1[<, +1]$ wrt $\mathrm{FO}^2[<, +1]$
- F_1 over trees
- A couple of specialized questions about decidability of some extensions of $\mathrm{F}_1[<, +1]$