The logical strength of Büchi's decidability theorem

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Motivation

The theory of automata over infinite words is interesting

- for model-checking
  - can model LTL, CTL, MSO, ...
- because of the word "infinite"
  - the *mysteries* of infinity
  - results are much less elementary than in the finite case

Motivating question

How much axiomatic strength is required to develop this theory?

Büchi automata and MSO

Acceptance condition of Büchi automata

$w \in \Sigma^{\mathbb{N}}$ accepted by $\mathcal{A}$

$$\Leftrightarrow$$

$\exists \rho$ run of $\mathcal{A}$ over w with $\rho(n) \in F$ for infinitely many n

Monadic Second-Order logic ≡ logic of order restricted to unary predicates (read as sets)

Typical statement of this language:

"Y contains an infinite set" $\equiv \exists X \, (X \subseteq Y \;\wedge\; \forall n \;\exists k \;(n \leq k \;\wedge\; k \in X))$

Büchi's theorem

Theorem [Büchi 62]

$\mathrm{MSO}(\mathbb{N}, \leq)$ is decidable.

The proof hinges on several automata constructions

- recognizing the union of two recognizable languages for $\vee$
- projections for $\exists$
- complementation for $\neg$

Complementation

The non-elementary step is complementation of Büchi automata.

There are two popular ways of accomplishing this

- direct complementation
  - original solution by Büchi
  - uses the infinite Ramsey theorem ($RT^2_{<\infty}$)
- go through determinization . . .
  - determinization itself is nontrivial
  - uses weak König's lemma (WKL_0)

Questions

- is one of those "harder" than the other?
- how to formalize it?

Reverse Mathematics

A convenient framework to formalize these questions is given by the programme of *Reverse Mathematics*.

Methodology

Study theorems of interest in weak subsystems of second-order arithmetic (Z_2)

Typical statements ressemble "Over RCA_0...

- Bolzano-Weierstrass theorem is equivalent to König's lemma (ACA_0)"
- Gödel's completeness theorem is equivalent to WKL_0"

Subsystems of Z_2

Typical subsystems of second-order arithmetic restrict the shape of the formulae in

- induction schemes
- comprehension schemes
  - an arbitrary formula cannot be considered a second-order object

The base theory RCA_0

We will be working with the weak theory RCA_0.

RCA_0 (recursive comprehension axiom) restricts Z_2 to

- Σ^0_1-induction (Σ^0_1-IND)
- Δ^0_1-comprehension

Intuition

A Σ^0_1 formula $\varphi(n)$ corresponds to recursively enumerable sets

- (relative to φ's parameters)
- hence Δ^0_1 corresponds to decidability
- RCA_0's minimal model is (ω, Dec)

$\mathsf{RT}^2_{<\infty}$ and WKL_0 in Reverse Mathematics

As one might suspect, $\mathsf{RT}^2_{<\infty}$ and WKL_0 are nontrivial in this framework

- (ω, Dec) does not satisfy either $\mathsf{RT}^2_{<\infty}$ or WKL_0
- over RCA_0, WKL_0 and $\mathsf{RT}^2_{<\infty}$ are known to be incomparable

$\leadsto$ what is going on in Büchi's theorem is not obvious

- is determinization essentially harder than complementation…
- …or are $\mathsf{RT}^2_{<\infty}$ and WKL_0 an overkill?

$\mathsf{RT}^2_{<\infty}$ and WKL_0 in Reverse Mathematics

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- is determinization essentially harder than complementation…
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Our main theorem

Theorem

Over RCA_0, the following are equivalent:

- decidability of $\mathsf{MSO}(\mathbb{N})$[1]
- complementing Büchi automata
- Σ^0_2-induction (Σ^0_2-IND)

Moreover, each of the above imply soundness of determinization

[1] Technically, of any fragment with fixed quantifier alternation ≥ 5.

Comments

Moral

Σ^0_2-IND characterizes the logical strength of Büchi's theorem.

- Σ^0_2-IND is orthogonal to WKL_0 and strictly weaker than $\mathsf{RT}^2_{<\infty}$
- We can instantiate this result in any model of RCA_0.
  - for instance, it means that $\mathsf{MSO}(\omega, \mathcal{P}(\omega)) \equiv \mathsf{MSO}(\omega, \mathrm{Dec})$
- Σ^0_2-IND seems to be a minimal prerequisite
  - If φ is Δ^0_1, "There are finitely many n such that $\varphi(n)$" is Σ^0_2
- This is in stark contrast with the situation with tree automata
  - see *How unprovable is Rabin's decidability theorem?* (L. A. Kołodziejczyk, H. Michalewski, 2015)

To sum up, over RCA_0...

WKL_0 $\mathsf{RT}^2_{<\infty}$

$\mathsf{RT}^2_{<\infty}$ and WKL_0: strong, incomparable

To sum up, over RCA_0...

WKL_0 $\qquad\qquad$ $\mathsf{RT}^2_{<\infty}$

$$\begin{array}{ccccc} & & \Sigma^0_2\text{-}\mathsf{IND} & & \\ & \Nearrow & & \Searrow & \\ \mathrm{MSO}(\mathbb{N}) & & \Leftarrow & & \text{Compl.} \end{array}$$

$$\Downarrow$$

Determinization

$\mathsf{RT}^2_{<\infty}$ and WKL_0: strong, incomparable

Σ^2_0-IND matches our needs

To sum up, over RCA_0...

$$
\begin{tikzcd}
\mathsf{WKL}_0 & & & \mathsf{RT}^2_{<\infty} \arrow[d, Rightarrow] \\
 & & \Sigma^0_2\text{-}\mathsf{IND} \arrow[r, Leftrightarrow] \arrow[dr, Rightarrow] & \mathsf{ART}^2_{<\infty} \arrow[d, Rightarrow] \\
 & \mathrm{MSO}(\mathbb{N}) \arrow[ur, Rightarrow] & & \mathrm{Compl.} \arrow[ll, Rightarrow] \\
 & & \mathrm{Determinization} &
\end{tikzcd}
$$

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Colorings valued in finite monoids

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& \mathrm{MSO}(\mathbb{N}) \arrow[ur, Rightarrow] \arrow[dl, Rightarrow] & & \text{Compl.} \arrow[ll, Rightarrow] \arrow[dl, Rightarrow] & \\
\mathsf{BWKL}_0 & & \text{Determinization} & &
\end{tikzcd}
$$

$\mathsf{RT}^2_{<\infty}$ and WKL_0: strong, incomparable

Σ^2_0-IND matches our needs

Colorings valued in finite monoids

Trees have width bounded by $|Q|$

Open questions

- figure out whether determinization alone implies Σ^0_2-IND
- make sure Σ^0_2-IND is enough to show the soundness of other determinization procedures
  - we studied Muller-Schupp; Safra's construction would be a good target
  - another interesting target would be determinization in terms of Wilke algebras
- other problems concerning automata over infinite words could be calibrated
  - the uniformization theorem
    - "for a given automaton $\mathcal{A}$ such that $\forall X\, \exists Y\, (\mathcal{A}$ accepts $X \otimes Y)$, there exists $\mathcal{B}$ such that $\forall X\, \exists! Y\, ($both $\mathcal{A}$ and $\mathcal{B}$ accept $X \otimes Y)$"

Thanks for your attention!

$$
\begin{tikzcd}
\mathsf{WKL}_0 \arrow[ddd, Rightarrow] & & & & \mathsf{RT}^2_{<\infty} \arrow[d, Rightarrow] \\
& & \Sigma^0_2\text{-}\mathsf{IND} \arrow[rr, Leftrightarrow] \arrow[dr, Rightarrow] & & \mathsf{ART}^2_{<\infty} \arrow[dl, Rightarrow] \\
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\mathsf{BWKL}_0 & & & \mathrm{Determinization} &
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