

# The logical strength of Büchi's decidability theorem

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September 1, 2016

# Motivation

The theory of automata over infinite words is interesting

- for model-checking
  - can model LTL, CTL, MSO, ...
- because of the word “infinite”
  - the *mysteries* of infinity
  - results are much less elementary than in the finite case

## Motivating question

How much axiomatic strength is required to develop this theory?

# Büchi automata and MSO

## Acceptance condition of Büchi automata

$w \in \Sigma^{\mathbb{N}}$  accepted by  $\mathcal{A}$

$\Leftrightarrow$

$\exists \rho$  run of  $\mathcal{A}$  over  $w$  with  $\rho(n) \in F$  for infinitely many  $n$

*Monadic Second-Order logic*  $\equiv$  logic of order restricted to unary predicates (read as sets)

Typical statement of this language:

“ $Y$  contains an infinite set”  $\equiv \exists X (X \subseteq Y \wedge \forall n \exists k (n \leq k \wedge k \in X))$

# Büchi's theorem

Theorem [Büchi 62]

$\text{MSO}(\mathbb{N}, \leq)$  is decidable.

The proof hinges on several automata constructions

- recognizing the union of two recognizable languages for  $\vee$
- projections for  $\exists$
- complementation for  $\neg$

# Complementation

The non-elementary step is complementation of Büchi automata.

There are two popular ways of accomplishing this

- direct complementation
  - original solution by Büchi
  - uses the infinite Ramsey theorem ( $RT_{<\infty}^2$ )
- go through determinization . . .
  - determinization itself is nontrivial
  - uses weak König's lemma ( $WKL_0$ )

## Questions

- is one of those “harder” than the other?
- how to formalize it?

# Reverse Mathematics

A convenient framework to formalize these questions is given by the programme of *Reverse Mathematics*.

## Methodology

Study theorems of interest in weak subsystems of second-order arithmetic ( $Z_2$ )

Typical statements resemble “Over  $RCA_0$ ...

- Bolzano-Weierstrass theorem is equivalent to König's lemma ( $ACA_0$ )”
- Gödel's completeness theorem is equivalent to  $WKL_0$ ”

# Subsystems of $Z_2$

Typical subsystems of second-order arithmetic restrict the shape of the formulae in

- induction schemes
- comprehension schemes
  - an arbitrary formula cannot be considered a second-order object

# The base theory $\text{RCA}_0$

We will be working with the weak theory  $\text{RCA}_0$ .

$\text{RCA}_0$  (recursive comprehension axiom) restricts  $\text{Z}_2$  to

- $\Sigma_1^0$ -induction ( $\Sigma_1^0$ -IND)
- $\Delta_1^0$ -comprehension

## Intuition

A  $\Sigma_1^0$  formula  $\varphi(n)$  corresponds to recursively enumerable sets

- (relative to  $\varphi$ 's parameters)
- hence  $\Delta_1^0$  corresponds to decidability
- $\text{RCA}_0$ 's minimal model is  $(\omega, \text{Dec})$



# $RT_{<\infty}^2$ and $WKL_0$ in Reverse Mathematics

As one might suspect,  $RT_{<\infty}^2$  and  $WKL_0$  are nontrivial in this framework

- $(\omega, \text{Dec})$  does not satisfy either  $RT_{<\infty}^2$  or  $WKL_0$
- over  $\text{RCA}_0$ ,  $WKL_0$  and  $RT_{<\infty}^2$  are known to be incomparable

$\leadsto$  what is going on in Büchi's theorem is not obvious

- is determinization essentially harder than complementation...
- ...or are  $RT_{<\infty}^2$  and  $WKL_0$  an overkill?

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- ...or are  $RT_{<\infty}^2$  and  $WKL_0$  an overkill? *THEY ARE*

# Our main theorem

## Theorem

Over  $\text{RCA}_0$ , the following are equivalent:

- decidability of  $\text{MSO}(\mathbb{N})^1$
- complementing Büchi automata
- $\Sigma_2^0$ -induction ( $\Sigma_2^0$ -IND)

Moreover, each of the above imply soundness of determinization

<sup>1</sup> Technically, of any fragment with fixed quantifier alternation  $\geq 5$ .

# Comments

## Moral

$\Sigma_2^0$ -IND characterizes the logical strength of Büchi's theorem.

- $\Sigma_2^0$ -IND is orthogonal to  $WKL_0$  and strictly weaker than  $RT_{<\infty}^2$
- We can instantiate this result in any model of  $RCA_0$ .
  - for instance, it means that  $MSO(\omega, \mathcal{P}(\omega)) \equiv MSO(\omega, Dec)$
- $\Sigma_2^0$ -IND seems to be a minimal prerequisite
  - If  $\varphi$  is  $\Delta_1^0$ , "There are finitely many  $n$  such that  $\varphi(n)$ " is  $\Sigma_2^0$
- This is in stark contrast with the situation with tree automata
  - see *How unprovable is Rabin's decidability theorem?* (L. A. Kołodziejczyk, H. Michalewski, 2015)

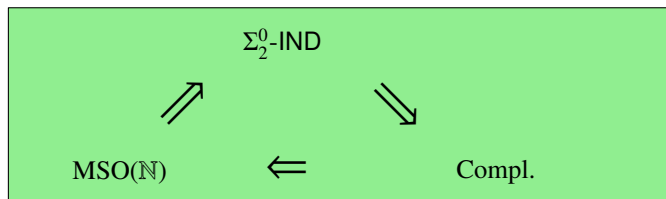
# To sum up, over $\text{RCA}_0 \dots$

$\text{WKL}_0$

$\text{RT}_{<\infty}^2$

$\text{RT}_{<\infty}^2$  and  $\text{WKL}_0$ : strong, incomparable

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 $\text{WKL}_0$ 
 $\text{RT}_{<\infty}^2$ 


Determinization

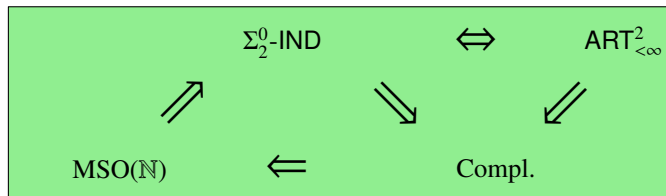
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$\Sigma_0^2\text{-IND}$  matches our needs

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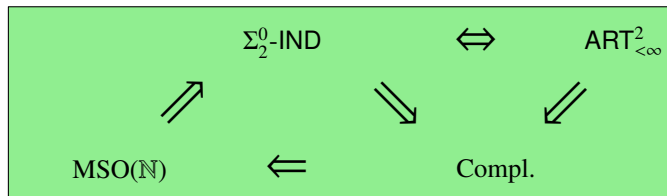


Determinization

$\text{RT}_{<\infty}^2$  and  $\text{WKL}_0$ : strong, incomparable

Colorings valued in finite monoids

$\Sigma_0^2$ -IND matches our needs

To sum up, over  $\text{RCA}_0 \dots$  $\text{WKL}_0$  $\text{RT}_{<\infty}^2$  $\text{BWKL}_0$ 

Determinization

$\text{RT}_{<\infty}^2$  and  $\text{WKL}_0$ : strong, incomparable

$\Sigma_2^0\text{-IND}$  matches our needs

Colorings valued in finite monoids

Trees have width bounded by  $|Q|$



# Open questions

- figure out whether determinization alone implies  $\Sigma_2^0$ -IND
- make sure  $\Sigma_2^0$ -IND is enough to show the soundness of other determinization procedures
  - we studied Muller-Schupp; Safra's construction would be a good target
  - another interesting target would be determinization in terms of Wilke algebras
- other problems concerning automata over infinite words could be calibrated
  - the uniformization theorem
    - “for a given automaton  $\mathcal{A}$  such that  $\forall X \exists Y (\mathcal{A} \text{ accepts } X \otimes Y)$ , there exists  $\mathcal{B}$  such that  $\forall X \exists ! Y$  (both  $\mathcal{A}$  and  $\mathcal{B}$  accept  $X \otimes Y$ )”

# Thanks for your attention!

