Counting in Team Semantics

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Objective

Bring together two threads of research in logic.

- Counting constructs in logic.
- Logics of dependence and independence, based on team semantics.

We introduce two kinds of counting operators for team semantics.

- forking
- counting quantifiers

Explore the expressive power of logics extended with these operators.

Main result: Inclusion logic extended with counting quantifiers is equal to fixed-point logic with counting.

Counting in Logics

Counting is an important task in numerous applications, but with a somewhat delicate relationship to logic.

Already elementary cardinality statements are not expressible in FO, or its extension with fixed-points (LFP).

$$\text{EVEN} := \{\mathfrak{A} : \text{the structure } \mathfrak{A} \text{ has an even number of elements}\}$$

This motivated fixed-point logic with counting (FPC).

FPC is of fundamental importance in descriptive complexity theory.

Reference in the quest for a logic that captures PTIME. It already captures PTIME on many important classes of structures.

Team Semantics

Hodges (1997): provided a model-theoretic (compositional) semantics for the independence friendly logic IF.

Dependency statements such as x *depends on* y or that x *and* y *are independent* do not make much sense for single assignments.

Formulae $\psi(x_1, \dots, x_k)$ are not evaluated on a single assignment $s : \{x_1, \dots, x_k\} \to A$, but on a set of such assignments, called a team.

$$\mathfrak{A} \models_X \varphi(\bar{x}) : \varphi \text{ is true in the structure } \mathfrak{A} \text{ for the team } X$$

Väänänen (2007): formalized dependencies as atomic statements on teams (rather than by annotations of quantifiers).

For a team X of assignments $s : V \to A$:

Dependence: $=(\bar{x}, \bar{y}) \iff (\forall s \in X)(\forall s' \in X)(s(\bar{x}) = s'(\bar{x}) \to s(\bar{y}) = s'(\bar{y}))$

Inclusion: $(\bar{x} \subseteq \bar{y}) \iff (\forall s \in X)(\exists s' \in X)(s(\bar{x}) = s'(\bar{y}))$

Exclusion: $(\bar{x} \mid \bar{y}) \iff (\forall s \in X)(\forall s' \in X)(s(\bar{x}) \neq s'(\bar{y}))$

Independence: $(\bar{x} \perp \bar{y}) \iff X(\bar{x}\bar{y}) = X(\bar{x}) \times X(\bar{y})$

Here $X(\bar{x})$ is the set of all values $s(\bar{x})$ for $s \in X$.

Team Semantics Inductive Definition

- For $\psi(\bar{y}) \in \mathrm{FO} : \mathfrak{A} \models_X \psi(\bar{y}) \iff \mathfrak{A} \models_s \psi(\bar{y})$ for all $s \in X$
- $\mathfrak{A} \models_X \psi \wedge \varphi \iff \mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_X \varphi$
- $\mathfrak{A} \models_X \psi \vee \varphi \iff X = Y \cup Z$ such that $\mathfrak{A} \models_Y \psi$ and $\mathfrak{A} \models_Z \varphi$
- $\mathfrak{A} \models_X \forall y \psi \iff \mathfrak{A} \models_{X[y \mapsto A]} \psi$
- $\mathfrak{A} \models_X \exists y \psi \iff \exists F : X \to \mathcal{P}(A) \setminus \{\emptyset\}$ such that $\mathfrak{A} \models_{X[y \mapsto F]} \psi$

For sentences we define: $\mathfrak{A} \models \psi \iff \mathfrak{A} \models_{\{\emptyset\}} \psi$

Notice that we cannot reasonably replace $\{\emptyset\}$ by $\emptyset$, since the empty team satisfies all formulae: $\mathfrak{A} \models_\emptyset \psi$ for all ψ.

From Team Semantics to Tarski Semantics

A team X of assignments $s : V \to A$ can be viewed as relation $X \subseteq A^{|V|}$.

A formula $\psi(\bar{x})$ in a logic with team semantics and vocabulary τ, can be translated to a Σ_1^1-sentence $\psi^*(X)$ of vocabulary $\tau \cup \{X\}$ such that

$$\mathfrak{A} \models_X \psi(\bar{x}) \Longleftrightarrow (\mathfrak{A}, X) \models \psi^*(X).$$

The expressive power of a logic L with team semantics can be understood by describing the equivalent fragment $\mathrm{F} \subseteq \Sigma_1^1$.

Team Semantics and Existential Second-Order Logic

1. Exclusion logic and dependence logic capture precisely the downwards closed Σ^1_1-properties of teams (sentences in which the team predicate appears only negatively).

2. FO + inclusion + exclusion $\equiv$ independence logic $\equiv \Sigma^1_1$. Thus, all NP-properties of teams are definable in independence logic.

3. **Galliani, Hella (2013)**: Inclusion logic $\equiv$ LFP, so it captures PTIME on all ordered finite structures:

   For every $\varphi(\bar{z}) \in \mathrm{FO}(\subseteq)$ one can construct a formula $\psi(X, \bar{z}) \in$ posGFP, and vice versa, such that, for all $\mathfrak{A}$ and all X

   $$\mathfrak{A} \models_X \varphi(\bar{z}) \Longleftrightarrow (\mathfrak{A}, X) \models \forall \bar{z}(X\bar{z} \rightarrow \psi(X, \bar{z})).$$

   Hence the maximal team satisfying φ coincides with $\mathbf{gfp}(\psi)$. On finite structures the posGFP-fragment coincides with LFP.

Counting in Team Semantics

To add counting in a meaningful way, consider two-sorted structures

$$\mathfrak{A}^* = \mathfrak{A} \cup (\omega, <, +, \cdot, 0, |A|).$$

What are appropriate counting constructs for logics with team semantics ?

Investigate their expressive and closure properties.

Can the Galliani-Hella theorem be extended to a characterization of fixed-point logic with counting (FPC) by a logic with team semantics ?

Counting in Team Semantics

The second-order features of teams lead to rich variety of potential counting constructs. Consider the majority quantifier $My\,\varphi(y,\bar{v})$.

$$G \models_s My\,Exy \iff s(x) \text{ is adjacent to at least half of the nodes in } G$$

A team X of assignments defines a set $U = X(x) \subseteq V$ of nodes.
Different possible definitions for $G \models_X MyExy$:

1. every $u \in U$ is adjacent to at least half of the nodes in G
2. every $u \in U$ is adjacent to same set of at least half of the nodes in G
3. U (as a set) is adjacent to at least half of the nodes in G

With different requirements for F, all three correspond to semantic rule

$$\mathfrak{A} \models_X My\,\varphi(y,\bar{v}) \iff \mathfrak{A} \models_{X[y\mapsto F]} \varphi \text{ for some } F : X \to \mathcal{P}(A)$$

Counting in Team Semantics

We explore two counting mechanisms for team semantics:

Forking atoms:

$$\mathfrak{A}^* \models_X (\bar{x} \lessdot^{\geq \mu} y) \iff (\forall s \in X) |\{s'(y) : s(\bar{x}) = s'(\bar{x}), s' \in X\}| \geq s(\mu)$$

Count the number of values for y in the assignments that coincide on $\bar{x}$.
Forking atoms $\bar{x} \lessdot^{\leq \mu} y$ and $\bar{x} \lessdot^{= \mu} y$ are defined analogously.

Counting quantifiers:

$$\mathfrak{A}^* \models_X \exists^{\geq \mu} x \varphi \iff \mathfrak{A}^* \models_{X[x \mapsto F]} \varphi \text{ for } F : X \to \mathcal{P}(A) \text{ with } |F(s)| \geq s(\mu)$$

Extend each assignment $s \in X$ with at least $s(\mu)$ values for x.

All three variants of the majority quantifier for team semantics are definable via counting quantifiers and/or forking.

Forking and Dependence

Forking atoms:

$$\mathfrak{A}^* \models_X (\bar{x} \lhd^{\geq\mu} y) \iff (\forall s \in X)|\{s'(y) : s(\bar{x}) = s'(\bar{x}), s' \in X\}| \geq s(\mu)$$

Generalizes dependence: $=(\bar{x}, y) \;\equiv\; \bar{x} \lhd^{\leq 1} y \;\equiv\; \bar{x} \lhd^{=1} y$

Forking atoms do not take us out of Σ^1_1.

Closure properties:
$\bar{x} \lhd^{\leq\mu} y$ is downwards closed, but not closed under union of teams
$\bar{x} \lhd^{\geq\mu} y$ is closed under union of teams, but not downwards closed
$\bar{x} \lhd^{=\mu} y$ is neither downwards closed, nor closed under union of teams

Theorem. $\mathrm{FO}(\bar{x} \lhd^{\leq\mu} y) \;\equiv\; \mathrm{FO}(\mathrm{dep}) \;\subsetneq\; \mathrm{FO}(\bar{x} \lhd^{=\mu} y) \equiv \mathrm{FO}(\mathrm{indep})$.

Forking atoms are powerful. Classical notions of dependence and independence reappear in this setting. Forking leads to simple logical definitions of many natural NP-properties (Vertex Cover, Clique).

Counting Quantifiers and Inclusion

Counting quantifiers:

$$\mathfrak{A}^* \models_X \exists^{\geq\mu} x\varphi \iff \mathfrak{A}^* \models_{X[x\mapsto F]} \varphi \text{ for } F\colon X \to \mathcal{P}(A) \text{ with } |F(s)| \geq s(\mu)$$

The Galliani-Hella theorem can be extended for counting:

For first-order logic with inclusion atoms ($\bar{x}\bar{\mu} \subseteq \bar{y}\bar{\nu}$) and counting quantifiers $\exists^{\geq\mu}$, indeed, it holds that $\text{FO}(\subseteq, \exists^{\geq\mu}) \equiv \text{FPC}$.

Theorem. For every $\varphi(\bar{x}, \bar{\mu}) \in \text{FO}(\subseteq, \exists^{\geq\mu})$ one can construct a formula $\psi(X, \bar{x}\bar{\mu}) \in \text{FPC}$, and vice versa, such that, for all two-sorted structures $\mathfrak{A}^*$ and all X

$$\mathfrak{A}^* \models_X \varphi(\bar{x}, \bar{\mu}) \iff (\mathfrak{A}^*, X) \models \forall\bar{x}\bar{\mu}(X\bar{x}\bar{\mu} \to \psi(X, \bar{x}\bar{\mu})).$$

Hence the maximal team satisfying φ coincides with $\mathbf{gfp}(\psi)$.

We prove this with a game-based approach.

Threshold Safety Games and Game Interpretations

Threshold game: $\mathcal{G} = (V, E, \theta : V \to \omega)$

At position v, player 0 chooses set $X \subseteq vE$ such that $|X| \geq \theta(v)$ and player 1 selects $x \in X$ from where the game proceeds.

Player who cannot move loses. This means that Player 0 loses at all nodes v with $\delta(v) < \theta(v)$ and Player 1 loses at nodes v with $\theta(v) = 0$.

Threshold safety games: Infinite plays are won by Player 0

Threshold safety games $\mathcal{T}(\mathfrak{A}^*, \psi)$, with a certain trap condition, are model-checking games for both FPC and FO$(\subseteq, \exists^{\geq \mu})$.

Moreover, for every formula $\psi(\bar{x}, \bar{\mu})$ there is a first-order interpretation I_ψ that interprets the game $\mathcal{T}(\mathfrak{A}^*, \psi)$ in $\mathfrak{A}^*$.

Translation between Logics via Interpretation

Winning regions and trap conditions in threshold safety games are definable in FPC, and also in $\mathrm{FO}(\subseteq, \exists^{\geq\mu})$.

Translate any $\psi(\bar{x}, \bar{\mu})$ from $\mathrm{FO}(\subseteq, \exists^{\geq\mu})$ into FPC:

Take formula from FPC that defines necessary trap condition in threshold safety model-checking game for $\psi(\bar{x}, \bar{\mu})$.

Interpretation I_ψ translates this formula into another FPC formula, that is evaluated in $\mathfrak{A}^*$ (still describes the trap condition of the game).

Combining this with the definability of the relevant starting positions, we obtain an FPC-formula that is equivalent to $\psi(\bar{x}, \bar{\mu})$.

An analogous argument works for translation of FPC to $\mathrm{FO}(\subseteq, \exists^{\geq\mu})$.

Conclusion

Forking is powerful concept that expresses dependencies itself and can describe many properties more naturally than existing constructs.

Counting quantifiers in combination with inclusion logic capture fixed-point logic with counting.

Introduced threshold safety games as model checking games.

Framework of two-sorted structures and counting constructs relevant for multi-team semantics.

There are many possibilities for counting mechanisms over teams.