A Sequent Calculus for a Modal Logic on Finite Data Trees

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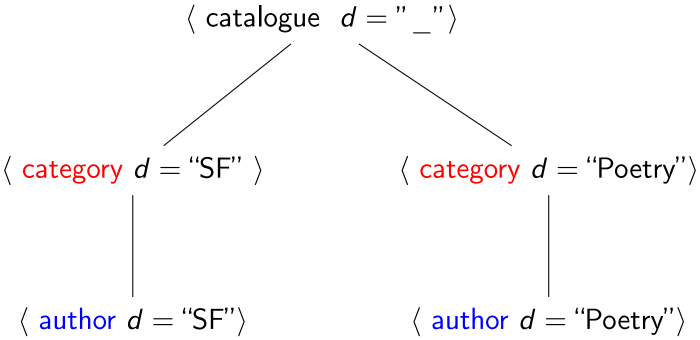
XML and XPath

- An XML tree :



- A request : $\Box($ category $\supset \Diamond_{=}$ author $)$

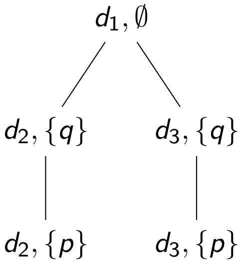
Validity problem for modal data logic

- Modal data logic : addressed by model theory and combinatorial algorithms (e.g. Figueira, 2010)
- Want to study it with proof theory
  - $\rightarrow$ witness for validity of a formula (proof tree) and a counter-example otherwise
  - $\rightarrow$ amenable to reasonable implementations avoiding worst-case complexity
  - $\rightarrow$ same complexity as combinatorial algorithms

Data tree

A *data tree* $\mathcal{A} = (A, \delta, \mathcal{V})$ is a tree $A = (T, <)$ where :

- T the set of nodes of A,
- $<$ the usual descendant relation on trees,
- $\delta\colon T \longrightarrow \mathbb{D}$ a function which associates a datum from a countable set $\mathbb{D}$ to a node of A,
- $\mathcal{V}\colon T \longrightarrow \mathcal{P}(\mathbb{A})$ a valuation which associates a set of atomic propositions to every node of A.



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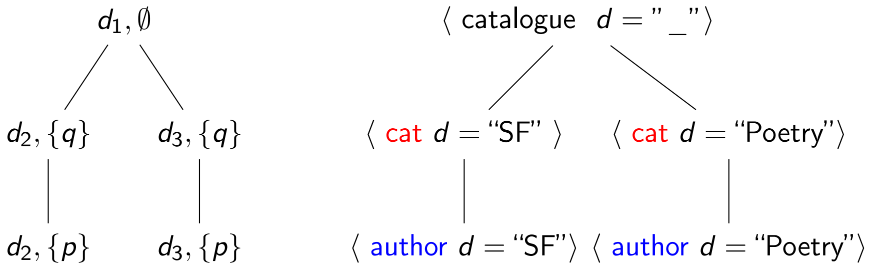

Modal Data Logic (DataGL)

Syntax

$\phi ::= a \mid \bot \mid \phi \supset \phi \mid \Box_{=}\phi \mid \Box_{\neq}\phi$,
where $a \in \mathbb{A}$ the set of atomic propositions.

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Semantic

Let $\mathcal{A} = (A, <, \delta, \mathcal{V})$ be a *finite data tree*.

- $t \models a$ iff $a \in \mathcal{V}(w)$.
- $t \not\models \bot$
- $t \models \phi \supset \psi$ iff if $t \models \phi$, then $t \models \psi$.
- $t \models \Box_{=}\phi$ iff $\forall t_1 < t$ s.t. $\delta(t_1) = \delta(t)$, then $t_1 \models \phi$.
- $t \models \Box_{\neq}\psi$ iff $\forall t_1 < t$ s.t. $\delta(t_1) \neq \delta(t)$, then $t_1 \models \psi$.

DataGL : a bimodal logic

Definition

Two relations $R^{=}$ and $R^{\neq}$:

$$\begin{cases} tR^{=}t_1 & \textit{iff} \quad t_1 < t \textit{ and } \delta(t_1) = \delta(t), \\ tR^{\neq}t_1 & \textit{iff} \quad t_1 < t \textit{ and } \delta(t_1) \neq \delta(t). \end{cases}$$

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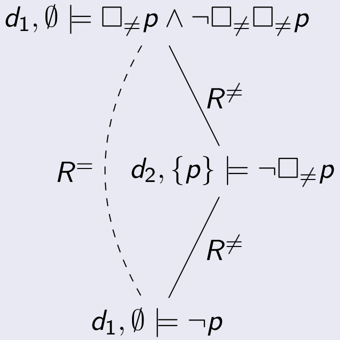
- $R^{=}$ transitive and irreflexive,
- $R^{\neq}$ irreflexive,
  and if $t_1 R^{\neq} t_2$ and $t_2 R^{\neq} t_3$, then $t_1 R^{\neq} t_3$ or $t_1 R^{=} t_3$.

DataGL : an example

Example

- $\Box_= p \supset \Box_= \Box_= p$ *is valid,*
- *but* $\Box_{\neq} p \supset \Box_{\neq} \Box_{\neq} p$ *is not.*



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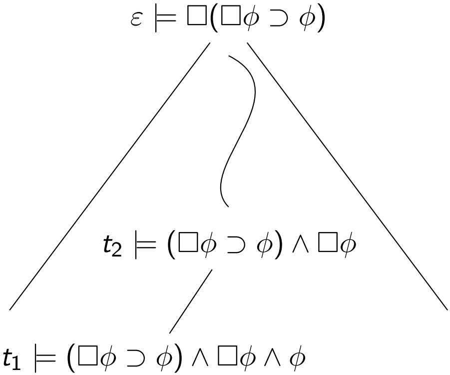

Avron's sequent calculus (1984)

Complete sequent calculus for GL
extending a standard propositional calculus with :

$$\frac{\Box\Gamma, \Gamma, \Box\phi \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\,\Box$$

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- Extraction of the information of $\Box\Gamma$ thanks to transitivity and irreflexivity,
- $\Box\phi$ is here to integrate axiom (L).

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$$\Big|\ R^=$$

$$t \models \Box_=\Gamma^= \wedge \Gamma^= \wedge \Box_{\neq}\Gamma^{\neq} \wedge \Box_=\phi$$

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  - $\rightarrow$ Furthermore, need to remember the previous hypotheses.
- Solution : Use an history $\mathcal{H}$ which remembers the hypotheses previously encountered.

History

The history $\mathcal{H}$ is a finite multi-set of sets of formulæ.
We will denote it $\mathcal{H} = H_0; \ldots; H_n$.

Rule for the modality $\Box_{\neq}$

- The two-premises case :

$$\dfrac{\begin{array}{l}\ldots \\ \ldots\end{array}}{\Box_{=} H_0^{=}, \Box_{\neq} H_0^{\neq}; \Box_{=} \Gamma^{=}, \Box_{\neq} \Gamma^{\neq} \Rightarrow \Box_{\neq} \psi} \; \Box_{\neq}$$

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$$\begin{tikzcd}
& \varepsilon \models \Box_= H_0^= \wedge \Box_{\neq} H_0^{\neq} \arrow[d, dash, "R^{\neq}"] \arrow[ddl, dash, dashed, bend right, "R^{\neq}"'] \\
& t \models \Box_= \Gamma^= \wedge \Box_{\neq} \Gamma^{\neq} \arrow[dl, dash, "R^{\neq}"] \\
t_1 \models H_0^{\neq} \wedge \Gamma^{\neq} &
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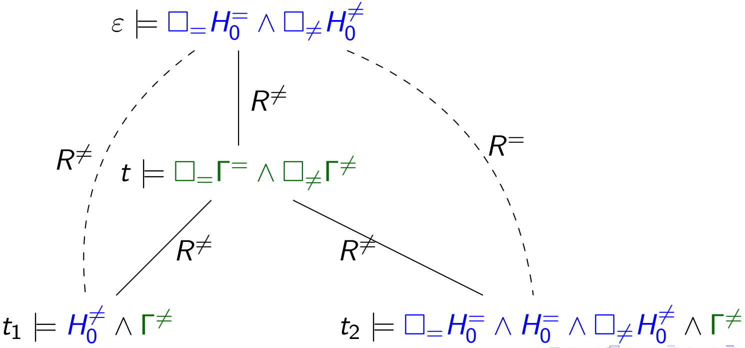

Example of a proof search

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  }\, \Box_{\neq}
}\, \Box
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Rule for $\Box_{\neq}$

$$\dfrac{\mathcal{H}; \Box_{=}\Gamma^{=}, \Box_{\neq}\Gamma^{\neq}, \Box_{\neq}\psi; H_i^{\neq}, \Gamma^{\neq} \Rightarrow \psi \qquad (\mathcal{H} \setminus H_i; \Box_{=}\Gamma^{=}, \Box_{\neq}\Gamma^{\neq}, \Box_{\neq}\psi; \Box_{=}H_i^{=}, H_i^{=}, \Box_{\neq}H_i^{\neq}, H_{j_{j\neq i}}^{\neq}, \Gamma^{\neq} \Rightarrow \psi)_i}{\mathcal{H}; \Box_{=}\Gamma^{=}, \Box_{\neq}\Gamma^{\neq} \Rightarrow \Box_{\neq}\psi} \Box_{\neq}$$

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- n premises $=$ n cases to consider, but 2 kind of cases :
  - $\rightarrow$ First premise : datum of the arrival node different from the one of all other nodes.
  - $\rightarrow$ Other premises : datum of the arrival node is the same than the one of a previous node.
- Extraction of the information depends on the case.

Adequation of the calculus

Soundness theorem

The sequent calculus is sound.

Completeness theorem

The sequent calculus is complete.

Properties of our calculus

Small proof property

Let ϕ a provable formula.
We can construct a proof tree of polynomial depth.

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Small model property

Let ϕ be a non-provable formula.
We can build a counter-model of polynomial depth.

Lower complexity bound

Chagrov & Rybakov, 2002

The validity problem in GL is PSPACE-hard.

Lower complexity bound

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The validity problem in GL is PSPACE-hard.

Theorem

The validity problem in DataGL is PSPACE-hard.

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Theorem

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Corollary

The validity problem in DataGL is PSPACE-complete.

Conclusion and prospects

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Questions

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