Polymorphic Game Semantics for Dynamic Binding

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Goals

Extend game semantics of generic polymorphism in three (related) directions:

- Dynamic Binding of terms (and types) to variables — i.e. determined at run-time, not statically.
- Block structure — block scoping of definitions and polymorphic copying of blocks of code.
- Prove definability and full abstraction results for different combinations of computational effects (control, state, ...).

Motivations

- Discover a good semantic theory (e.g. categorical model) for dynamic binding, which is common in practice (virtual methods, overloading, dynamic dispatch).
- Understand relationship between poymorphism and dynamic scope, which seems to exist in our model at a deep level.
- Leverage the concrete/algorithmic representation of polymorphism in our semantics towards model checking for a language with polymorphism, integer state, classes,...
- Extend the "intensional hierarchy" relating computational effects with constraints on strategies.

Simply Typed Game Semantics: Block Structure

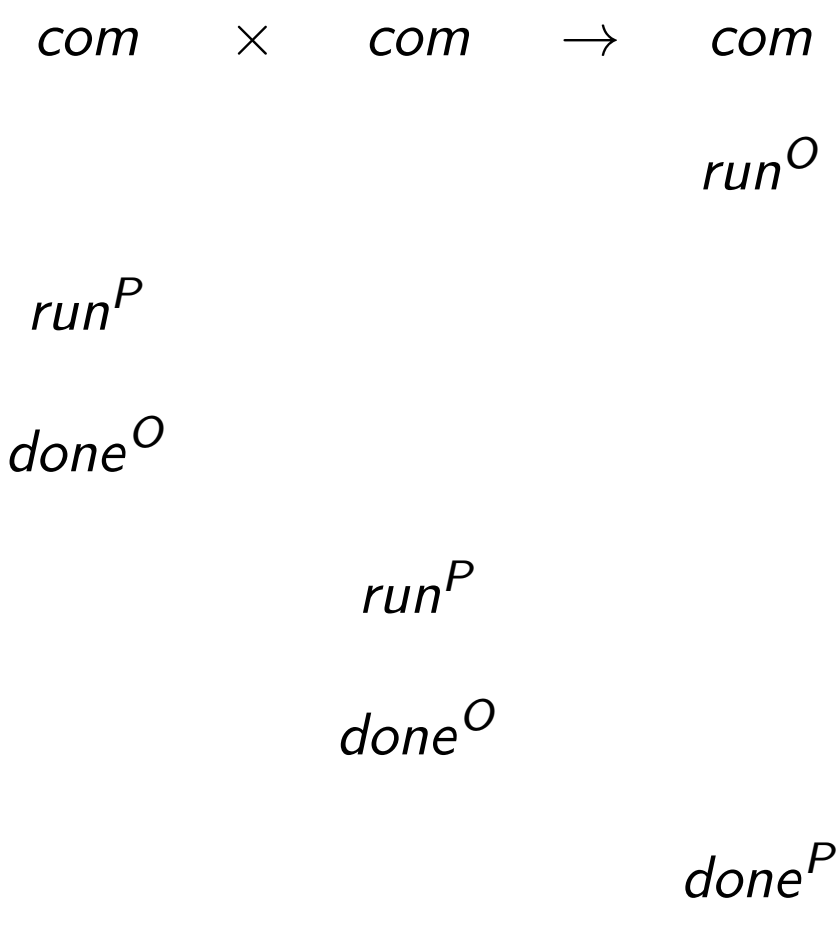
In game semantics of PCF (Hyland & Ong '00;Abrmasky, Jagadeesan & Malacaria '00) , Idealized Algol (Abramsky & McCusker '97), etc.:

- Types denote games for two players,
- who take turn to make moves,
- which are questions (procedure calls) or answers (returns)
- Questions and answers open and close evaluation of blocks of code.

E.g. — the type *com* of *commands* denotes the game with one question(*run* the command) with a single answer (*done*).

Simply Typed Game Semantics: Block Structure

Programs denote strategies for Player 2 ("Proponent") — e.g. sequential composition:



Second-Order Game Semantics

Generic Game Semantics of System F [L. 2010]
— question and answer moves denote type variables (occurring positively and negatively with respect to their binder)
E.g. $[\![\forall X.X \to X]\!]$ is the arena with one question, one answer.

$$\forall X.X \qquad \to \qquad X$$

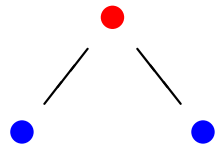
$$Q^O$$

$$A^P$$

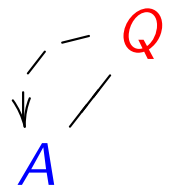
Questions and answers represent holes into which we can plug any other game tree — answering a question corresponds to playing copycat between these plugged-in games.

Example

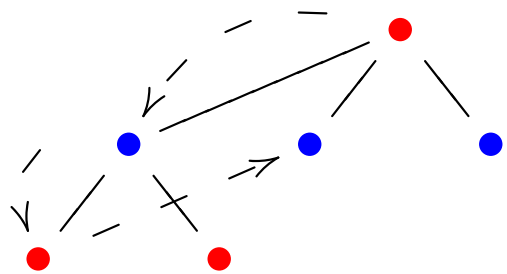
Plugging the arena $\forall Y.Y \to Y \to Y$:



into the identity strategy on $\forall X.X \to X$:



gives the identity on $(\forall Y.Y \to Y \to Y) \to \forall Y.Y \to Y \to Y$:



Subsuming Simple Games in Polymorphic Games

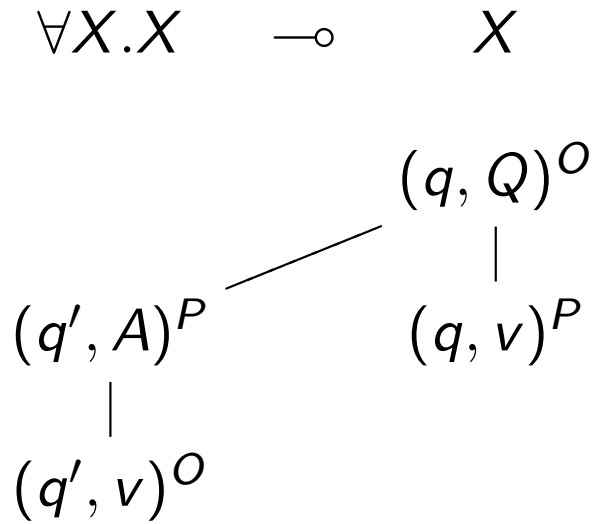
- Second order games subsume simply-typed ones via well-known System F encodings — e.g. $\forall X.X \to X$ denotes the same game as *com*,
- *But* we lose block structuring, for which we need basic types as atoms.
- Instantiation breaks static (and block) scoping in this model — to prove definability (fullness of the interpretation functor) and full abstraction (completeness w.r.t. observational equivalence), we need to add general references.

Polymorphic Game Semantics with Block Structure

Two levels of question-answer (bracketing):

- A move is either a simple question q or simple answer (q, v) (a simple question and a value).
- A simple question may be further labelled as a second-order question (q, Q) or second-order answer (q, A).
- Simple questions are endowed with enabling and conflict relations, which we use to build a model of (intuitionistic) linear type theory.

Example: NewInterpretation for $\forall X.(X \multimap X)$



Constraints: New and Old

Legal positions are determined by the unanswered (simple) questions, corresponding to the stack of open procedure calls — each move must be:

- enabled by a question on the stack if it is a simple question — only a function may call its arguments
- conflict-free with all moves on the stack if it is a simple question — only one call to a function at each nesting depth.
- a response to the question at the top of the stack if it is a second-order answer (well-bracketing)
- A simple answer copying the preceding value if and only if that is a simple answer which closes a second-order answer — polymorphic copycat condition.

Example

The only strategies on $\forall X.(X \multimap X)$ are $\{\varepsilon\}$ and the "generic identity" copycat $(\Sigma_{v \in \mathcal{V}}(q, Q)^O(q', A)^P(q', v)^O(q, v)^P)^*$.

$$
\begin{array}{ccc}
\forall X.X & \multimap & X \\
 & & (q, Q)^O \\
(q', A)^P & & \\
(q', v)^O & & \\
 & & (q, v)^P \\
 & & (q, Q)^O \\
(q', A)^P & & \\
(q', v')^O & & \\
 & & (q, v')^P \\
\vdots & &
\end{array}
$$

Dynamic Binding: Typing

Extend System F with a set of ground types B and a non-binding *let* operation.

$$\frac{\Theta \vdash \Gamma, T}{\Gamma, x{:}T, \Gamma' \vdash_\Theta x{:}T} \qquad \frac{\Gamma, x{:}S, \Gamma' \vdash_\Theta N{:}T \qquad \Gamma, x{:}S, \Gamma' \vdash_\Theta M{:}S}{\Gamma, x{:}S, \Gamma' \vdash_\Theta \mathit{let}\ x{=}M\ \mathit{in}\ N{:}T}$$

$$\frac{\Gamma, x{:}S \vdash_\Theta M{:}T}{\Gamma \vdash_\Theta \lambda x^S.M{:}S \to T} \qquad \frac{\Gamma \vdash_\Theta M{:}S \to T \qquad \Gamma \vdash_\Theta N{:}S}{\Gamma \vdash_\Theta M\ N{:}T}$$

$$\frac{\Gamma \vdash_{\Theta, X} M{:}T \qquad \Theta \vdash \Gamma}{\Gamma \vdash_\Theta \Lambda X.M{:}\forall X.T} \qquad \frac{\Gamma \vdash_\Theta M{:}\forall X.T \qquad \Theta \vdash S}{\Gamma \vdash_\Theta M\{S\}{:}T[S/X]}$$

Dynamic Binding: Operational Semantics

Programs are evaluated with respect to a stack of bindings — e.g.

$$\frac{M;\mathcal{E}\Downarrow 0;\mathcal{E}'}{\mathit{If}0\ M;\mathcal{E}\Downarrow\lambda xy.x;\mathcal{E}} \qquad \frac{M;\mathcal{E}\Downarrow\lambda x.M';\mathcal{E}'_a \quad M'[a/x];\mathcal{E}'_a,(a,N)\Downarrow C;\mathcal{E}''}{M\ N;\mathcal{E}\Downarrow C,\mathcal{E}''} \qquad \frac{M;\mathcal{E},(a,M),\mathcal{E}_a\Downarrow C;\mathcal{E}''}{a;\mathcal{E},(a,M),\mathcal{E}_a\Downarrow C;\mathcal{E}''}$$

$$\frac{M;\mathcal{E}\Downarrow n{+}1;\mathcal{E}'}{\mathit{If}0\ M;\mathcal{E}\Downarrow\lambda xy.y;\mathcal{E}} \qquad \frac{M;\mathcal{E}\Downarrow\Lambda X.M';\mathcal{E}' \quad M'[T/X];\mathcal{E}'\Downarrow C;\mathcal{E}''}{M\{T\};\mathcal{E}\Downarrow C,\mathcal{E}''} \qquad \frac{N;\mathcal{E},(a,M)\Downarrow C;\mathcal{E}'}{\mathit{let}\ a{=}M\ \mathit{in}\ N;\mathcal{E}\Downarrow C;\mathcal{E}'}$$

Examples

- Overriding of definitions: e.g. $\mathit{let}\ f = (\mathit{let}\ x = 1\ \mathit{in}\ M)\ \mathit{in}\ \mathit{let}\ x = 0\ \mathit{in}\ N$,
- Escape from static scope: $g : (R \to S \to S) \to R \to T) \vdash \lambda x^R.(g\ (\lambda y^T.\lambda z^S.\mathit{let}\ x = y\ \mathit{in}\ z))\ x : R \to T$ —the second argument to g may be captured from inside the scope of the first.
- No escape from block scope — e.g. $\lambda x^{nat}.\mathit{If}\ 0\ x\ \mathit{then}\ x\ \mathit{else}\ x$ and $\lambda x^{nat}.x$ are observationally equivalent (without integer state).

Dynamic Binding: Denotational Semantics

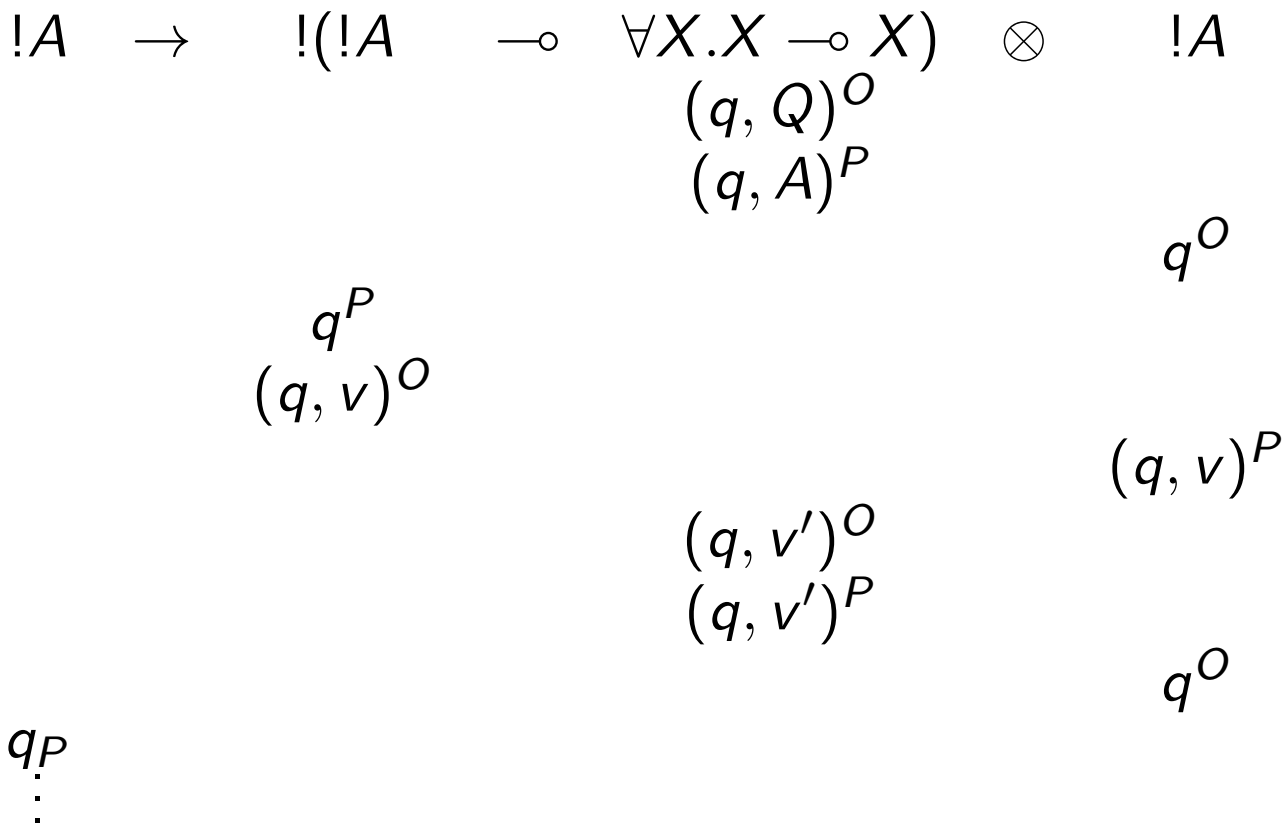
Like game semantics of references, exceptions, channels, locks, ... dynamically bound variables of type A may be considered as objects with two methods:

- definition (let) of "type" $!A \multimap \forall X.(X \multimap X)$
- invocation of type A

which we represent as projections from
$\S A = !(!A \multimap \forall X.(X \multimap X)) \otimes !A$.
So we interpret $x_1 : S_1, \ldots, x_n : S_n \vdash M : T$ as a morphism from $\S[\![S_1]\!] \otimes \ldots \otimes \S[\![S_n]\!]$ to $[\![T]\!]$.

Dynamic variable declaration strategy

To declare a new variable, compose with a strategy $dec_A : !A \to \S A$ which dynamically connects definitions to invocations:



Dynamic Binding: Categorical Semantics

To prove soundness of our semantics, we capture its categorical structure.

- $!A$ (ω-indexed copies of A, opened in order) is a minimal invariant (least fixed point) for $A \oslash \_$, where
- $\oslash$ is a monoidal action of our category of games on a subcategory $\mathcal{L}$ of strict maps, which is dual to $\multimap$ — i.e.

$$\frac{\mathcal{L}(A \oslash B, C)}{\mathcal{L}(A, B \multimap C)} \qquad \frac{\mathcal{L}(B \multimap A, C)}{\mathcal{L}(A, C \oslash B)}$$

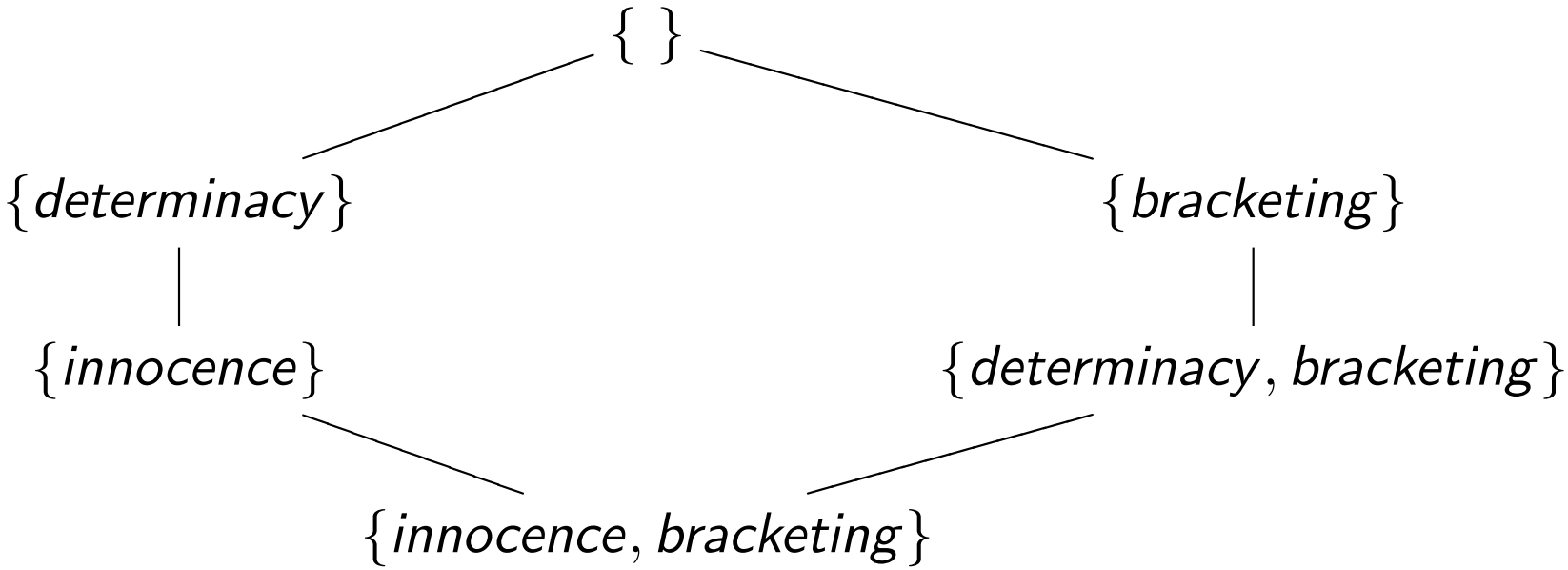
- We use these categorical properites to (corecursively) define declaration of dynamically bound variables, and prove its soundness.

Computational Effects

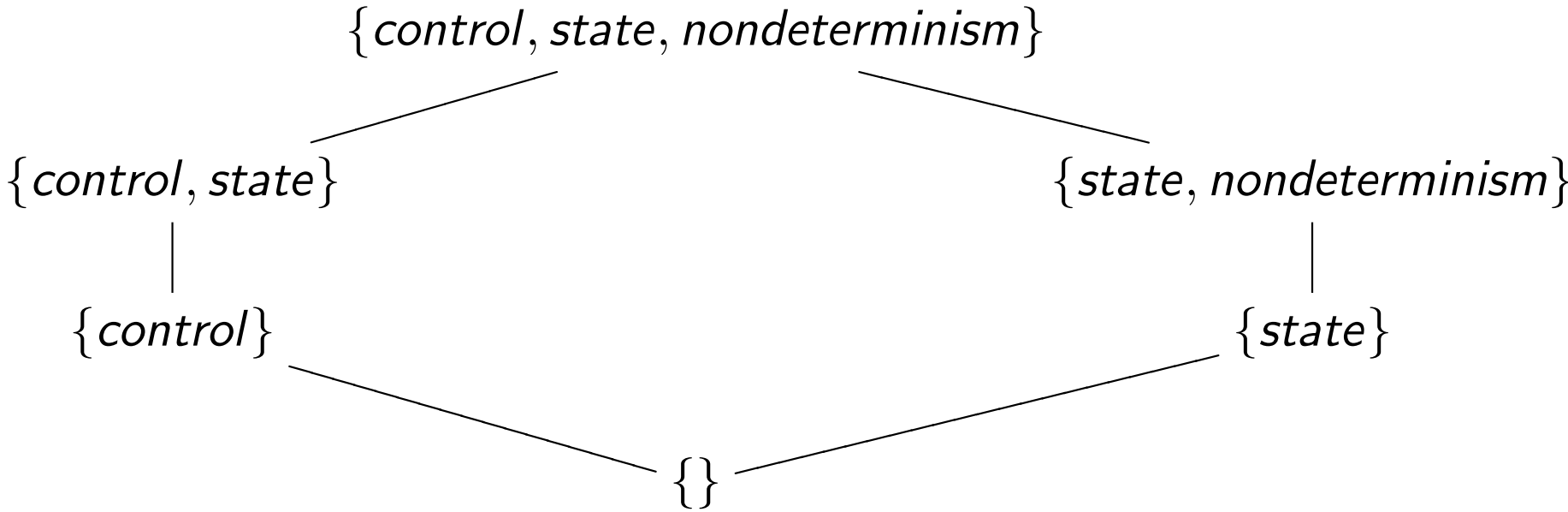
The "intensional hierarchy" on simple games relates:

- Constraints on strategies — used to identify PCF-definable strategies
- with computational effects — e.g. relaxing innocence gives a model of integer state (Idealized Algol).

Hierarchy of Constraints

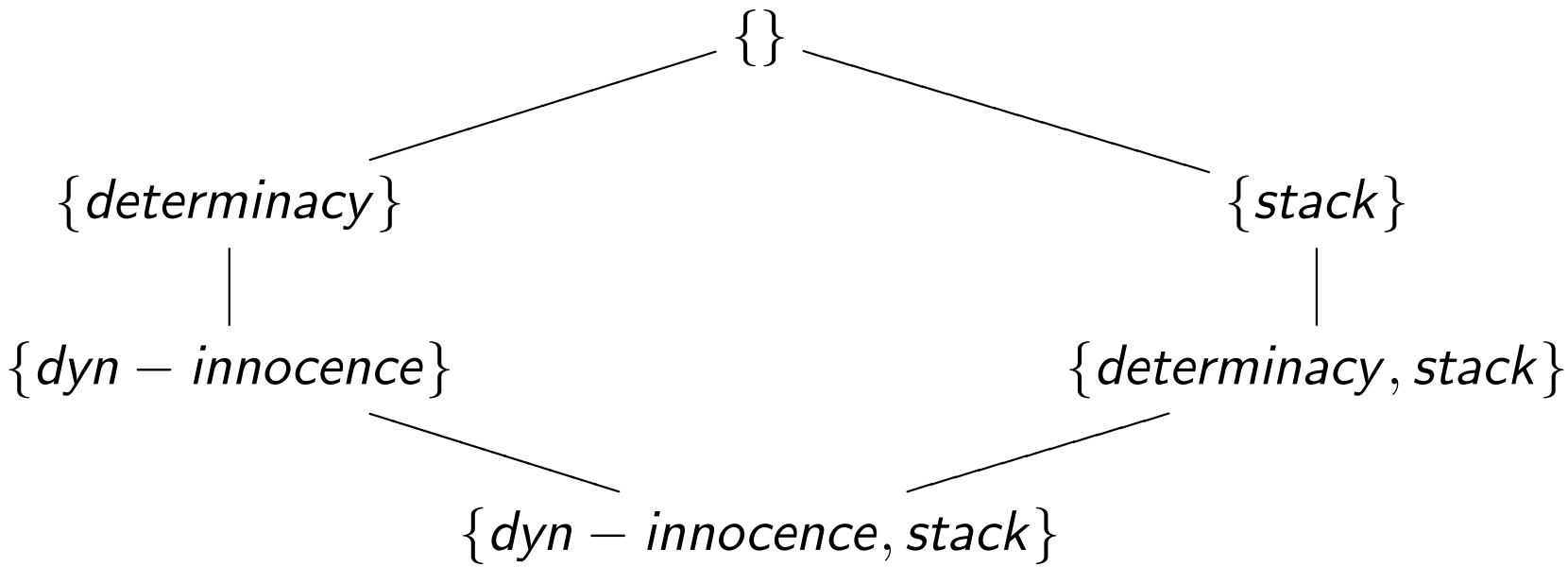


Hierarchy of Effects



An Intensional Hierarchy for Dynamic Binding

We define constraints on our model giving an intensional hierarchy of models with dynamic binding:



Definability and Full Abstraction

We prove

- PCF^2_d-definability for simply-typed strategies (by PCF-style decomposition).
- PCF^2_d-definability for strategies at all types, using a definable retraction $\forall X.T \trianglelefteq T[(\forall Y.Y \to Y)/X]$
- IA^2_d-definability and full abstraction for strategies which break dynamic innocence by an Idealized-Algol style factorization.

Questions?