Axioms for Modelling Cubical Type Theory in a Topos

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(joint work with Andrew Pitts)

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Homotopy type theory

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So what's a path from $a_0 : A$ to $a_1 : A$?

$$p : [0, 1] \to A$$
$$p\ 0 = a_0$$
$$p\ 1 = a_1$$

Cubical type theory [1]

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So let's add an interval $\mathbb{I}$ to model $[0,1]$:

$$r, s ::= 0 \mid 1 \mid i \mid r \wedge s \mid r \vee s \mid 1 - r$$

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Quoitented by the rules of a de Morgan algebra:

$$\mathbf{1} - (\mathbf{1} - r) = r$$
$$\mathbf{1} - 0 = 1$$
$$\mathbf{1} - 1 = 0$$
$$r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$$
$$\ldots$$

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Cubical type theory [1]

Now add a type of paths between $a_0, a_1 : A$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a_0 : A \qquad \Gamma \vdash a_1 : A}{\Gamma \vdash Path\ A\ a_0\ a_1}$$

with terms

$$\frac{\Gamma, i : \mathbb{I} \vdash a : A}{\Gamma \vdash \langle i \rangle a : Path\ A\ a[0/i]\ a[1/i]}$$

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Cubical type theory [1]

We have an interval object $\mathtt{I}$ which models the real interval $[0,1]$. From this we get:

$$
\begin{array}{rll}
\vdash a : A & & \text{(points)} \\
i : \mathtt{I} \vdash a : A & & \text{(lines)} \\
i : \mathtt{I}, j : \mathtt{I} \vdash a : A & & \text{(squares)} \\
i : \mathtt{I}, j : \mathtt{I}, k : \mathtt{I} \vdash a : A & & \text{(cubes)} \\
\dots & &
\end{array}
$$

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Partial terms

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These formulae specify a collection of corners, faces, edges, etc of an n-dimensional cube:

$$\varphi, \psi \ ::= \ \top \mid \bot \mid i = 0 \mid i = 1 \mid \varphi \wedge \psi \mid \varphi \vee \psi$$

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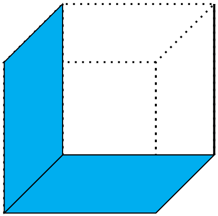
$$\Gamma, \varphi \vdash a : A$$

E.g.

$$\Gamma \triangleq i : \mathtt{I},\ j : \mathtt{I},\ k : \mathtt{I}$$

$$\varphi \triangleq (i = 0) \vee (j = 0) \vee (j = 1 \wedge k = 1)$$



A crash course in Kan filling

$$\Gamma, i : \mathbb{I} \vdash \mathit{fill}^i\ A\ [\varphi \mapsto u]\ a_0\ :\ A$$

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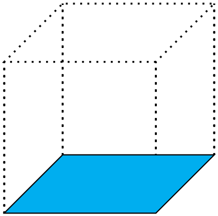

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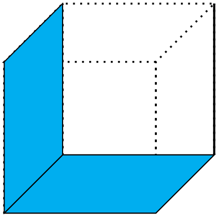

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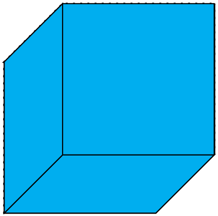

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- The face lattice $\mathbb{F}$ gives us a notion of partial elements and partial types.
- Types come equipped with a Kan composition/filling operation.
- The glueing construction gives a constructive interpretation of Voevodsky's univalence axiom.
- There is a (constructive) presheaf model.

Overview of our work

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A suggestion of Thierry Coquand

The internal type theory of a topos

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The internal type theory of a topos

- Standard interpretation of extensional type theory in a category with families (CwF) associated with any topos $\mathcal{E}$ (with families over $X \simeq \mathcal{E}/X$).
- The subobject classifier Ω becomes an impredicative universe of propositions with logical connectives, equality and quantifiers.
- The universal property of Ω gives rise to comprehension subtypes...

Comprehension subtypes

For any type $\Gamma \vdash A$ we can form comprehension subtypes:

$$\frac{\Gamma, x : A \vdash \varphi(x) : \Omega}{\Gamma \vdash \{x : A \mid \varphi(x)\}}$$

whose terms are those $t : A$ for which $\varphi(t)$ is provable.

Modelling partial terms/types

How do we model partial terms?

$$\Gamma, \varphi \vdash a : A$$

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We will make extensive use of these types in connection with partial elements.

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- $f : [\varphi] \to A$.

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We say that a partial element (φ, f) extends to $a : A$ if the following relation holds:

$$(\varphi, f) \nearrow a \triangleq \forall (u : [\varphi]).\ f\, u = a$$

Cofibrant propositions

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In our setting we will consider extending partial elements generated by a collection of cofibrant propositions (cf. the face lattice $\mathbb{F}$).

Cofibrant propositions

A collection of cofibrant propositions is a subobject $\mathtt{Cof} \rightarrowtail \mathbf{\Omega}$ such that:

$$\frac{i : \mathtt{I} \qquad e : \{0, 1\}}{(i = e) \in \mathtt{Cof}} \qquad \frac{\varphi \in \mathtt{Cof} \qquad \psi \in \mathtt{Cof}}{\varphi \vee \psi \in \mathtt{Cof}}$$

$$\frac{\varphi \in \mathtt{Cof} \qquad \varphi \Rightarrow (\psi \in \mathtt{Cof})}{\varphi \wedge \psi \in \mathtt{Cof}}$$

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Only needed for definitional/strong identity types

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$$\frac{\forall (i : \mathtt{I})\ (\varphi\, i \in \mathtt{Cof})}{(\forall (i : \mathtt{I})\, \varphi\, i) \in \mathtt{Cof}}$$

Required in the cubical sets model by Cohen et al. to get a univalent universe

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$$\frac{\varphi \in \mathtt{Cof} \qquad \varphi \Rightarrow (\psi \in \mathtt{Cof})}{\varphi \wedge \psi \in \mathtt{Cof}}$$

Cofibrant partial elements

A cofibrant partial element is a partial element (φ, f) such that φ is a cofibrant proposition. We write $\square A$ for the type of cofibrant partial elements of a type A.

$$\square A \triangleq (\varphi : \texttt{Cof}) \times ([\varphi] \to A)$$

Filling

We can now express a (simplified) notion of Kan filling in our internal type theory.

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The type of $\mathtt{0}$-filling structures for constant families, $\mathtt{Fill}\,\mathtt{0}$, is defined by

$$\begin{aligned}
&\mathtt{Fill}\,\mathtt{0}\,A \triangleq \\
&\quad (a : A) \\
&\quad (f_\varphi : \{f_\varphi : \Box(\mathtt{I} \to A) \mid f_\phi @ \mathtt{0} \nearrow a\}) \\
&\quad \to \\
&\quad \{g : \mathtt{I} \to A \mid f_\varphi \nearrow g \wedge g\,\mathtt{0} = a\}
\end{aligned}$$

Filling

$$
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$$

Filling

An element at one end

$$\mathtt{Fill}\,0\,A \triangleq$$
$$(a : A)$$
$$(f_\varphi : \{f_\varphi : \Box(\mathtt{I} \longrightarrow A) \mid f\,@\,0 \nearrow a\})$$
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Filling

An element at one end

A cofibrant (well behaved) partial path that agrees with the element

$$
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$$\mathtt{Fill}\,\mathtt{0}\,A \triangleq (a : A)\; (f_\varphi : \{f_\varphi : \square(\mathtt{I} \to A) \mid f\,@\,\mathtt{0} \nearrow a\}) \to \{g : \mathtt{I} \to A \mid f_\varphi \nearrow g \wedge g\,\mathtt{0} = a\}$$

A total path that extends the partial path and agrees with a at the end

Univalence

The univalence axiom states states that paths between two types, A and B, are essentially the same as equivalences between A and B.[2]

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In cubical type theory this is achieved using the glueing construction.

2. Stating this precisely is tricky.

Towards univalence

We internalise the glueing construction and show:

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but we do not yet have an internalisation of a univalent universe.

Agda development

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- Add an impredicative universe of mere propositions via a method due to Martin Escardo

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- Define a proof relevant version of $\{x : A \mid \phi\}$

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- Define a proof relevant version of $\{x : A \mid \phi\}$
- Postulate the axioms

Thanks for listening!

Summary:

- Any topos that satisfies the axioms will be a model of cubical type theory*

* without a universe object (for now).

"Axioms for Modelling Cubical Type Theory in a Topos"

Ian Orton and Andrew Pitts

Paper and Agda: http://www.cl.cam.ac.uk/~rio22/

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