

# Guarded Cubical Type Theory

## Path Equality for Guarded Recursion

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# In April at FoSSaCS...

We presented: **Guarded Dependent Type Theory**  
an extensional type theory with *guarded recursive types*.

Guarded recursion can, e.g., be used for

- programming and reasoning with coinductive types,
- building models of program logics  
(solving recursive domain equations synthetically).

# Guarded Recursive Types: Vocabulary

$\triangleright A$       the type of terms that will be  
 $A$  at the next tick of the clock

$\text{next } t$       the term  $t$  saved  
for the next tick of the clock

$\text{fix } x.t$       guarded fixed point of  $t$

$t \circledast u$       function application  
at the next tick of the clock

# Example: Streams

$\text{Stream} = \text{Nat} \times \triangleright \text{Stream}$

$\text{head} = \pi_1$

$\text{tail} = \pi_2$

$\text{zeros} = \text{fix } s. (0, s)$

$\text{map } f = \text{fix } m. \lambda s. (f (\text{head } s), m \circledast (\text{tail } s))$

# Example: Y Combinator

$$\text{Rec}_A = \triangleright \text{Rec}_A \rightarrow A$$

$$Y : (\triangleright A \rightarrow A) \rightarrow A$$

$$Y = \lambda f. \Delta (\text{next } \Delta)$$

where

$$\Delta : \triangleright \text{Rec}_A \rightarrow A$$

$$\Delta = \lambda x. f (x \otimes (\text{next } x))$$

# Going Dependent

$$\frac{\Gamma \vdash t : \triangleright(A \rightarrow B) \quad \Gamma \vdash u : \triangleright A}{\Gamma \vdash t \otimes u : \triangleright B}$$

$$\frac{\Gamma \vdash t : \triangleright((x : A) \rightarrow B) \quad \Gamma \vdash u : \triangleright A}{\Gamma \vdash t \otimes u : \triangleright ?}$$

# Going Dependent

$$\frac{\Gamma \vdash t : \triangleright(A \rightarrow B) \quad \Gamma \vdash u : \triangleright A}{\Gamma \vdash t \circledast u : \triangleright B}$$

$$\frac{\Gamma \vdash t : \triangleright((x : A) \rightarrow B) \quad \Gamma \vdash u : \triangleright A}{\Gamma \vdash t \circledast u : \triangleright[x \leftarrow u]. B}$$

Delayed substitutions

# Going Dependent

$$\frac{\Gamma \vdash t : \triangleright(A \rightarrow B) \quad \Gamma \vdash u : \triangleright A}{\Gamma \vdash t \otimes u : \triangleright B}$$

$$\frac{\Gamma \vdash t : \triangleright((x : A) \rightarrow B) \quad \Gamma \vdash u : \triangleright A}{\Gamma \vdash \text{next } [t' \leftarrow t, u' \leftarrow u]. t' u' : \triangleright[x \leftarrow u]. B}$$

Delayed substitutions



# Löb Induction

Guarded fixed points on identity type:

$$\frac{\Gamma, x : \triangleright(t = u) \vdash p : t = u}{\Gamma \vdash \text{fix } x. p : t = u}$$

This is powerful in combination with the term:

$$\text{com} : \triangleright(t = u) \rightarrow \text{next } t = \text{next } u,$$

to prove e.g. bisimilar streams equal:

$$\vdash \text{fix } x. \dots \text{com } x \dots : \text{map id zeros} = \text{zeros}$$

# Canonicity Loss

$\text{com} : \triangleright(t = u) \rightarrow \text{next } t = \text{next } u$

if “=” is intensional Martin-Löf identity type,  
“com” comes at the cost of *canonicity*:

$\vdash \text{fix } x. \dots \text{com } x \dots : \text{map id zeros} = \text{zeros}$

but

$\vdash \text{refl} \not= \text{map id zeros} = \text{zeros}$

# Identity Types

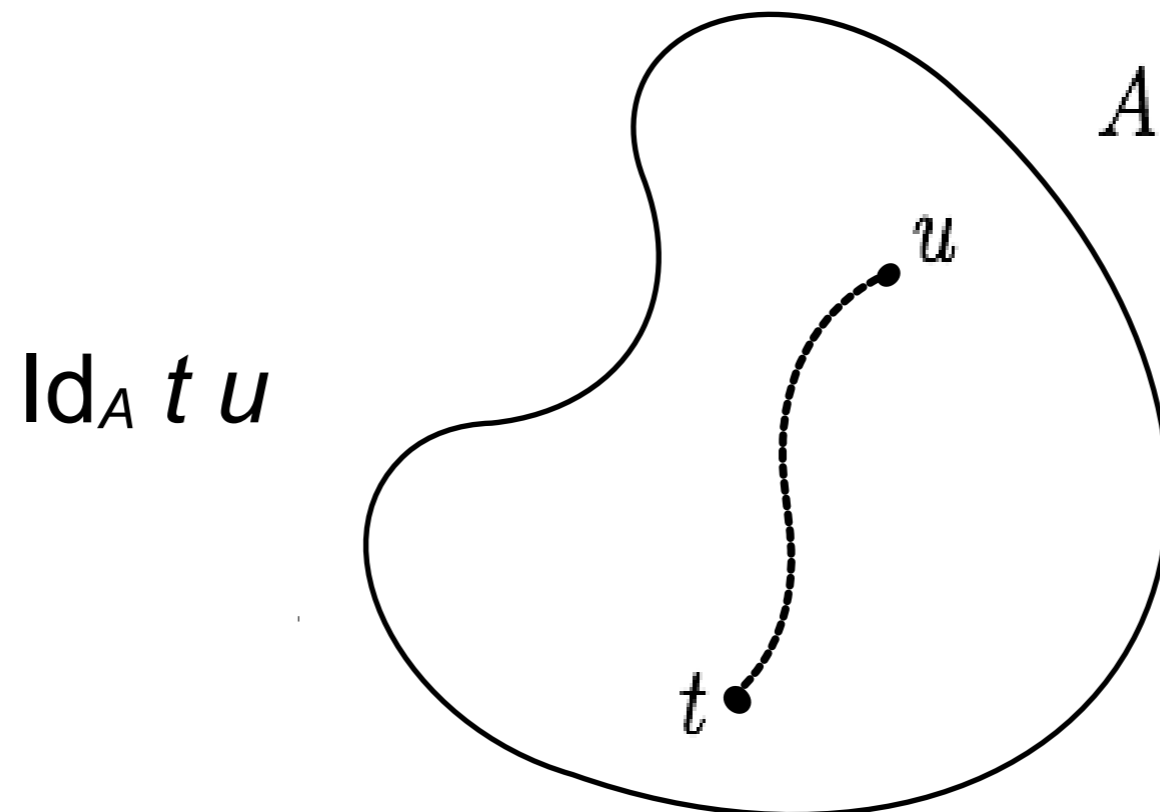
We want:

- a dependent type theory with
- guarded recursive types,
- canonicity,
- decidable type-checking.

So we need another notion of identity than the usual intensional or extensional identity types.

# Cubical Type Theory

Dependent type theory influenced by homotopy type theory: Types are **spaces**, identities are **paths**.



(Cohen, Coquand, Huber, Mörtberg 2016)

# Cubical Type Theory

We have a special *interval object*  $\mathbb{I}$ , and equalities are like functions from  $\mathbb{I}$  into a type:

$$\frac{\Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle t : \mathbf{Id}_A t[0/i] t[1/i]}$$

$$\frac{\Gamma \vdash p : \mathbf{Id}_A t u \quad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash pr : A} \quad \begin{array}{l} p 0 = t \\ p 1 = u \end{array}$$

# Cubical Type Theory

## Theorem:

- Consistency & soundness (w.r.t. *cubical sets model*)

## Conjectures:

- Canonicity (<https://arxiv.org/abs/1607.04156>)
- Decidable type-checking

## Implementation:

<https://github.com/mortberg/cubicaltt>

# Guarded Cubical Type Theory

We propose **Guarded Cubical Type Theory**, a combination of GDTT and CTT.

Paths and guarded recursive types play well together.  
Compare:

$$\text{funext} : ((a : A) \rightarrow \text{Id } B (f a) (g a)) \rightarrow \text{Id } (A \rightarrow B) f g$$
$$\text{funext} = \lambda p. \langle i \rangle \lambda a. p a i$$
$$\text{com} : \triangleright(\text{Id } A t u) \rightarrow \text{Id } (\triangleright A) (\text{next } t) (\text{next } u)$$
$$\text{com} = \lambda p. \langle i \rangle \text{next } [x \leftarrow p]. x i$$

# Unfolding Fixed Points

Obvious source of non-termination:

$$\text{fix } x.t = t[\text{next}(\text{fix } x.t)/x]$$

We have a *path equality* instead of definitional equality

$$\frac{\Gamma \vdash r : \mathbb{I} \quad \Gamma, x : \triangleright A \vdash t : A}{\Gamma \vdash \text{dfix}^r x.t : \triangleright A}$$

“delayed fixed point”

along with

$$\text{dfix}^1 x.t = \text{next} (t[\text{dfix}^0 x.t/x])$$

We write:  $\text{fix } x.t = t[\text{dfix}^0 x.t/x]$



# Guarded Cubical

Prototype type-checker at

<https://github.com/hansbugge/cubicaltt/tree/gcubical>

with examples.

# Semantics

**Lemma.**  $\mathcal{E}$  presheaf topos with non-trivial internal De Morgan algebra with the disjunction property  $\Rightarrow$  fibrant types of  $\mathcal{E}$  models CTT (*without* glueing and universe)

**Lemma.**  $\mathbb{C}$  category with initial object  $\Rightarrow$  fibrant types of  $\mathbf{PSh}(\square \times \mathbb{C})$  models CTT (*with* glueing and universe).  
( $\square$ : category of cubes).

**Theorem.** The fibrant types of  $\mathbf{PSh}(\square \times \omega)$  models GCTT. Especially:  $\triangleright A$  is fibrant.

# Conclusion and Future Work

## **We have**

- Consistent type theory – GCTT
- Model –  $\text{PSh}(\square \times \omega)$
- Implementation and examples

## **Future work**

- Algorithmic equality (decidable type checking)
- Canonicity
- Multiple clocks and clock quantification (real coinduction)