Extending Homotopy Type Theory with Strict Equality

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Reedy fibrant semi-simplicial types

$$\mathsf{Semi}_1 :\equiv \mathsf{Type}$$

Reedy fibrant semi-simplicial types

$$\begin{aligned} \mathrm{Semi}_2 :\equiv\ & (X_0 : \mathrm{Type}) \\ \times\ & (X_1 : X_0 \to X_0 \to \mathrm{Type}) \end{aligned}$$

Reedy fibrant semi-simplicial types

$$
\begin{aligned}
\mathrm{Semi}_3 :\equiv\ & (X_0 : \mathrm{Type}) \\
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\times\ & (X_2 \ :\ (x_0, x_1, x_2 : X_0) \\
& \qquad \to X_1(x_0, x_1) \to X_1(x_1, x_2) \to X_1(x_0, x_2) \\
& \qquad \to \mathrm{Type})
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Reedy fibrant semi-simplicial types

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\times\ & \cdots
\end{aligned}
$$

Semi-simplicial types

Functors $\Delta_+^{\mathrm{op}} \to \mathsf{Type}$:

- $X : \mathbb{N} \to \mathsf{Type}$;
- $(-)^* : \{n, m : \mathbb{N}\} \to \Delta(n, m) \to X_m \to X_n$;
- functor laws: $\sigma^*(\tau^*(x)) = (\tau \circ \sigma)^*(x)$, $\mathrm{id}^*(x) = x$;

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- $\ldots$

Infinite structures

- Simplicial or semi-simplicial types
- $(\infty, 1)$-categories and functors
- General eliminator for truncations
- Universes as models of type theory
- Internalising "types as ∞-groupoids"
- Giving a general definition of *higher inductive type*

Different infinite structures are related

- Semi-simplicial types $\Longrightarrow$ $(\infty, 1)$-categories
- $(\infty, 1)$-categories $\Longrightarrow$ semi-simplicial types
- Internal model $\Longrightarrow$ semi-simplicial types
- ...

Towards a solution

- Main idea: internalise judgemental equality
- First proposed by Voevodsky as HTS
- Models readily available

A two-level theory

Fibrant types

- Π, Σ, Id, 1, 0, $+$, $\mathbb{N}$, inductive types
- Higher inductive types
- Univalent universes

$\xRightarrow{\text{coerce}}$

Strict types or _pretypes_

- Π, Σ, Id, 1, 0, $+$, $\mathbb{N}$, inductive types
- Uniqueness of identity proofs
- Function extensionality
- Non-univalent universes

- Shared contexts and substitutions
- Coercion preserves Π, Σ and 1 *strictly*

Semi-simplicial types (revisited)

Functors $\Delta_+^{\mathrm{op}} \to \mathsf{Type}$:

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No coherence required

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\times {} & \cdots
\end{aligned}
$$

$$
\mathsf{Semi}_{n+1} :\equiv (X : \mathsf{Semi}_n) \times (M_n(X) \to \mathsf{Type})
$$

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$$\mathsf{Semi}_{n+1} :\equiv (X : \mathsf{Semi}_n) \times (M_n(X) \to \mathsf{Type})$$

$M_n(X)$ is defined using strict equality, but it can be shown to be fibrant

Where to go from here

- Implementation in existing proof assistants
- Conservativity of the two-level theory over its fibrant fragment
- Internal theory of semi-simplicial types
- Semi-Segal types as notions of $(\infty, 1)$-categories