Definability of Cai-Fürer-Immerman Problems in Choiceless Polynomial Time

Wied Pakusa, *Svenja Schalthöfer*, Erkal Selman

CSL 2016

Origin: Database theory

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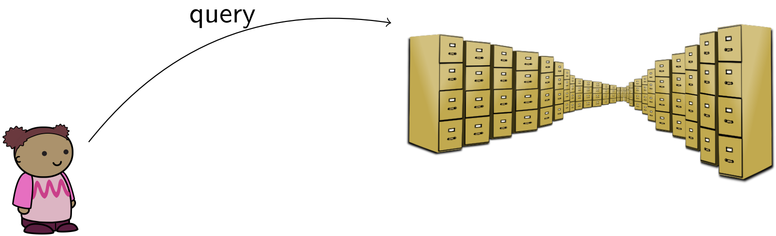

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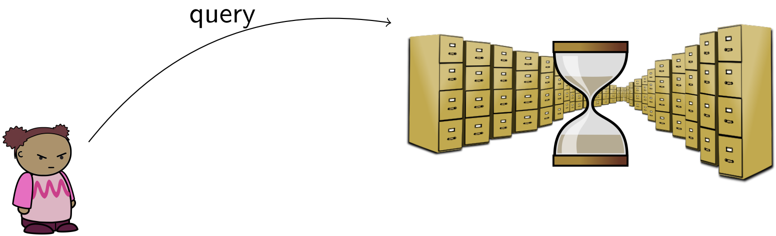

The most important problem in finite model theory

Question (Chandra, Harel 1982)

Is there a database query language expressing exactly the efficiently computable queries?

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Question (Gurevich 1988)

Is there a logic capturing PTIME?

What is a logic capturing PTIME?

Informal definition: A logic L captures $\textsc{Ptime}$ if it defines precisely those properties of finite structures that are decidable in polynomial time:

1. For every sentence $\psi \in L$, the set of finite models of ψ is decidable in polynomial time.
2. For every $\textsc{Ptime}$-property S of finite structures, there is a sentence $\psi \in L$ such that $S = \{\mathfrak{A} \in \mathsf{Fin} : \mathfrak{A} \models \psi\}$.

The most important problem in finite model theory

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Is there a database query language expressing exactly the efficiently computable queries?

Question (Gurevich 1988)

Is there a logic capturing $\textsc{Ptime}$?

Theorem (Fagin)

$\exists\mathrm{SO}$ captures NP.

Candidates for a logic capturing PTIME

PTIME

$\subsetneq$

FO   cannot define transitive closure

Candidates for a logic capturing PTIME

PTIME

$\subsetneq$

FP captures PTIME on ordered structures

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FO cannot define transitive closure

Candidates for a logic capturing PTIME

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$\rotatebox{90}{$\subsetneq$}$

FPC — captures PTIME on many interesting classes of structures.

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Candidates for a logic capturing PTIME

$\textsc{Ptime}$

$\subseteq \quad \subseteq$

$\mathrm{FP} + \mathsf{rk} \qquad \mathrm{CPT} + \mathrm{C}$

$\subsetneq \quad \subsetneq$

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Theorem (Dawar,Richerby,Rossman)

The Cai-Fürer-Immerman query over ordered graphs is CPT*-definable.*

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The logic: Choiceless Polynomial Time

Iterated creation of hereditarily finite sets,
polynomial resource bounds

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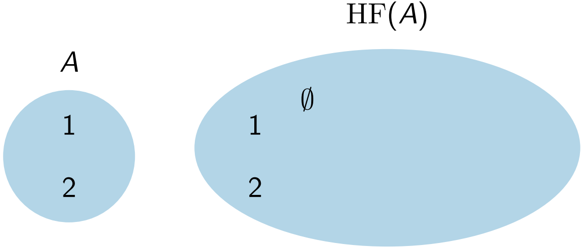
Hereditarily finite sets



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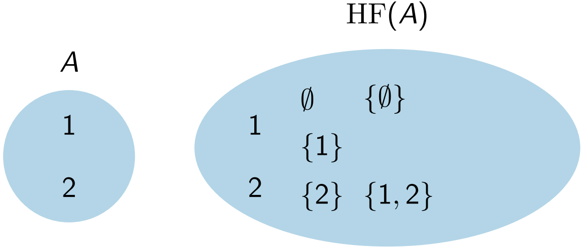

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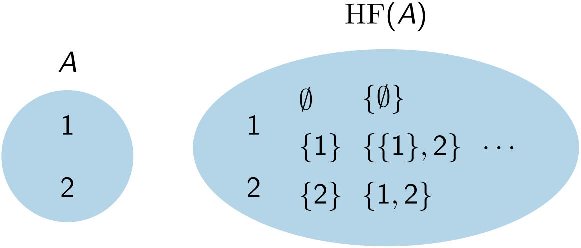

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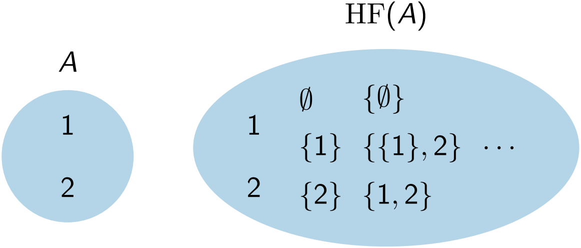

Operations

- Set-theoretic operations ($\emptyset$, $\in$, $\cup$,...), boolean connectives
- Comprehension terms: $\{s(x) : x \in t : \varphi(x)\}$

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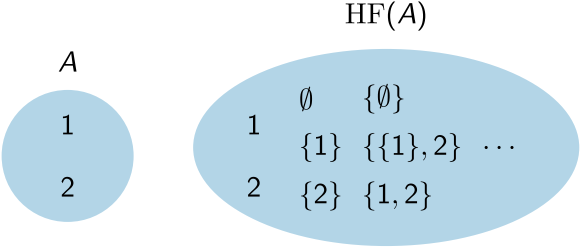

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- $\mathrm{P\small{TIME}}$-computable
- not FPC-definable
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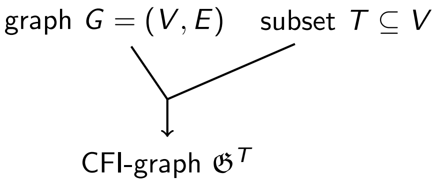

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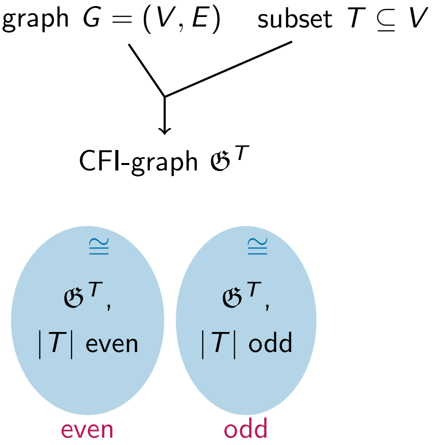

- PTIME-computable
- not FPC-definable
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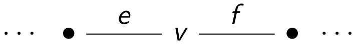
Cai-Fürer-Immerman graphs: construction

Graph $\mathfrak{G}$



vertex v

$\rightsquigarrow$ gadget v^*

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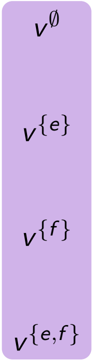
$$\cdots \quad \bullet \overset{e}{\text{———}} v \overset{f}{\text{———}} \bullet \quad \cdots$$



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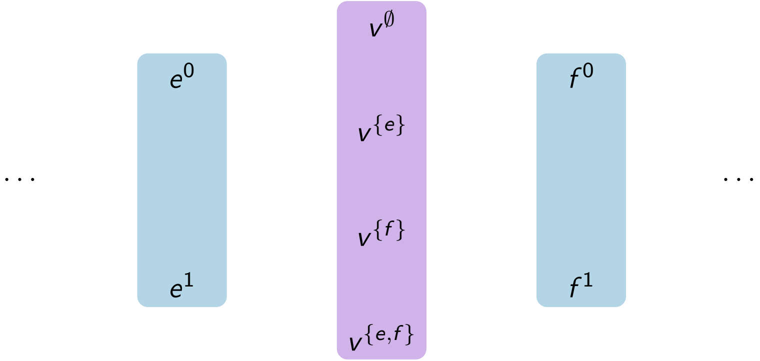

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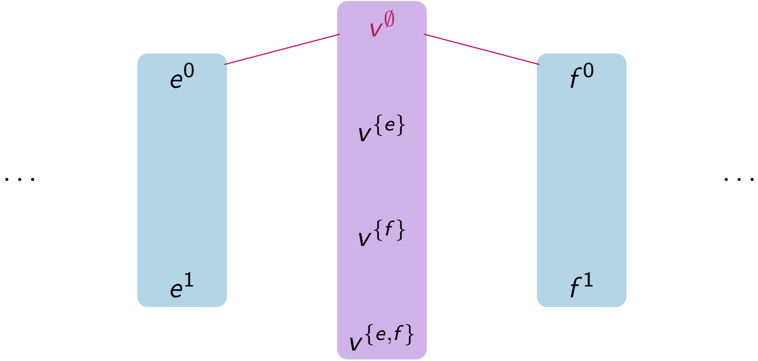

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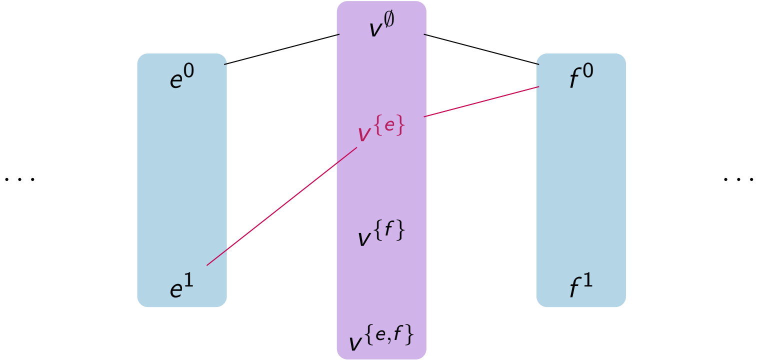

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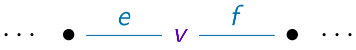

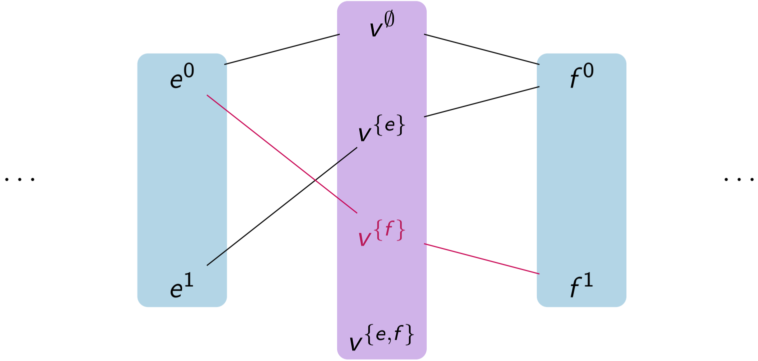

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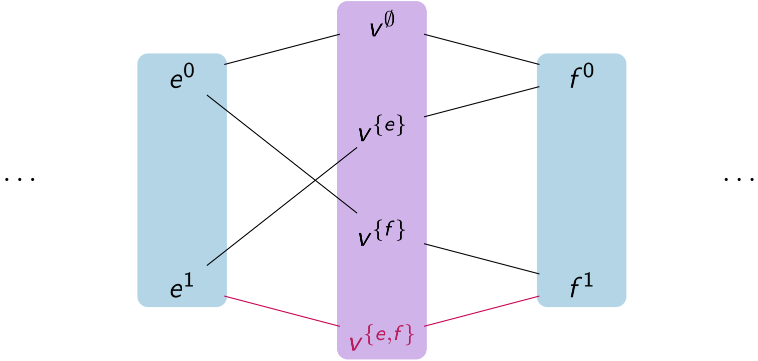

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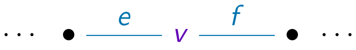
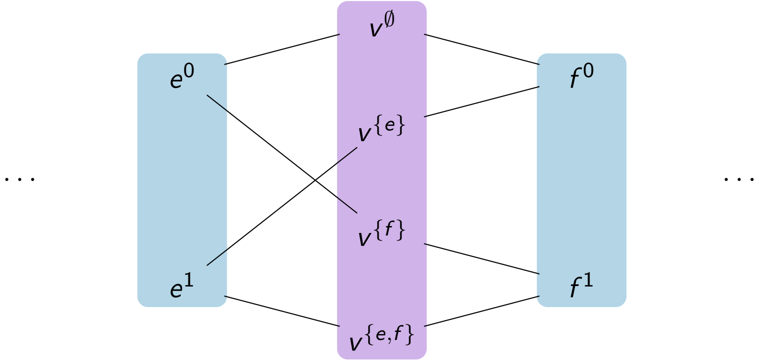
Graph $\mathfrak{G}^T$

vertex v
$\rightsquigarrow$ gadget v^*





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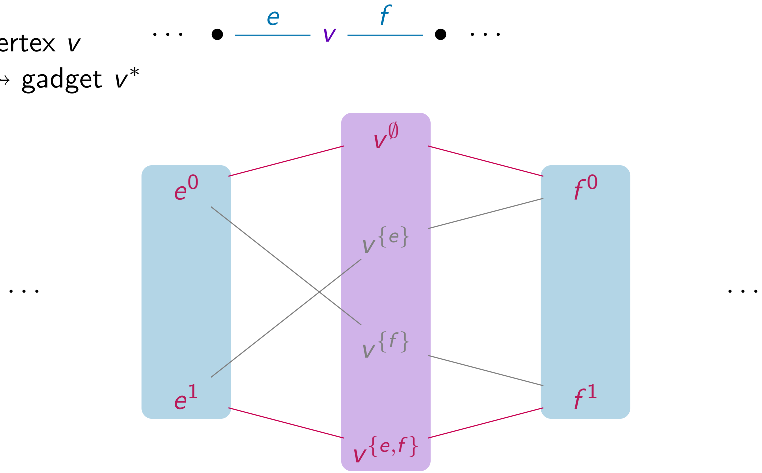

even gadget, $v \notin T$

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Graph $\mathfrak{G}^T$

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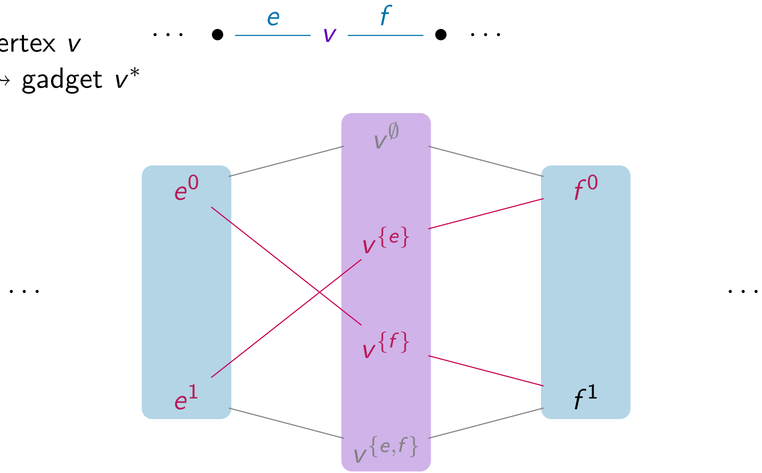
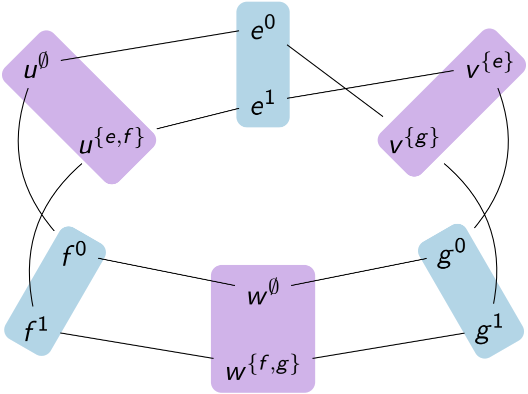


odd gadget, $v \in T$

Cai-Fürer-Immerman graphs: example



Theorem

The CFI query over graphs with logarithmic colour classes is CPT*-definable.*

Theorem

The CFI query over graphs with large degree is CPT*-definable using only sets of bounded rank.*

Theorem

The CFI query over complete graphs is not CPT*-definable without using set-like objects.*

Corollary

$\approx$ *-free* $\mathrm{PIL} \not\equiv \mathrm{CPT}[\mathrm{rk} \leq k]$

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Use:

Theorem (Dawar,Richerby,Rossman)
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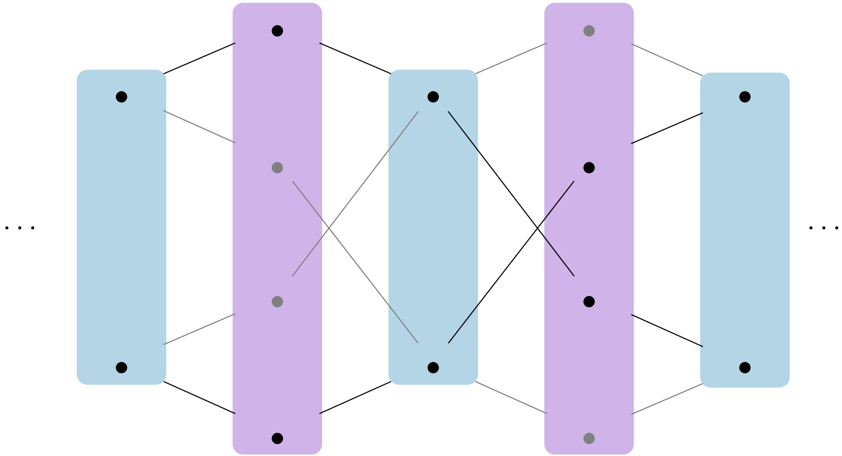
Computing the parity of CFI graphs

Easy PTIME procedure

1. Label edge gadgets
2. Count odd vertex gadgets

Assigning edge labels



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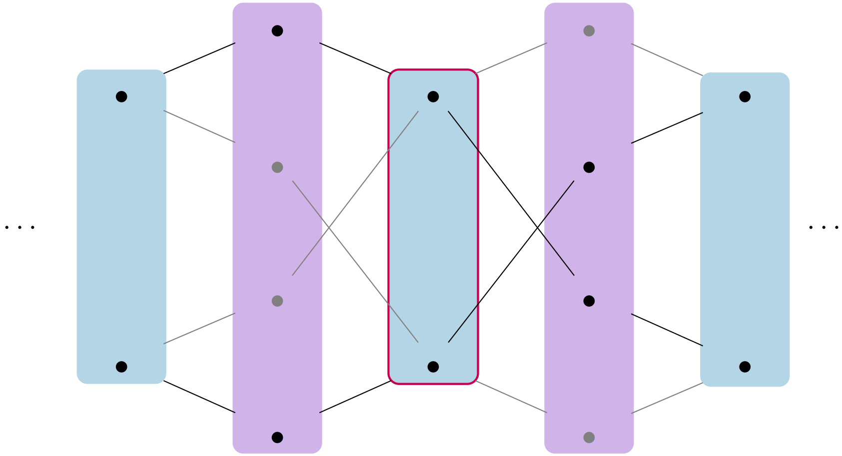

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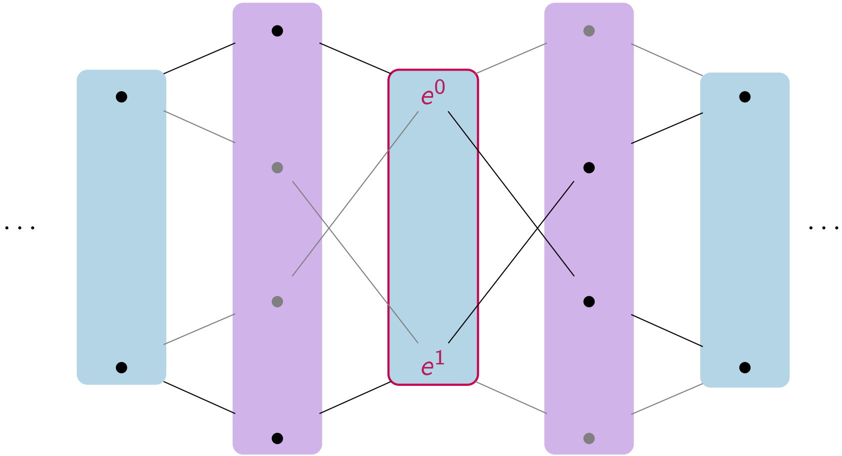

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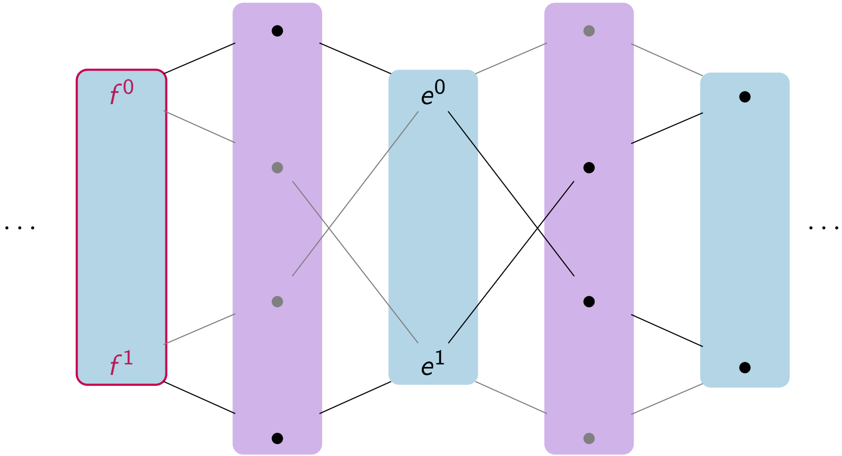

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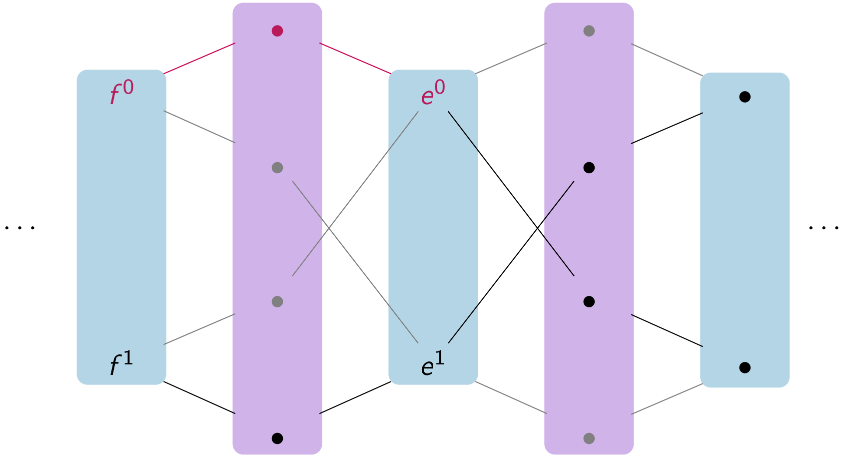

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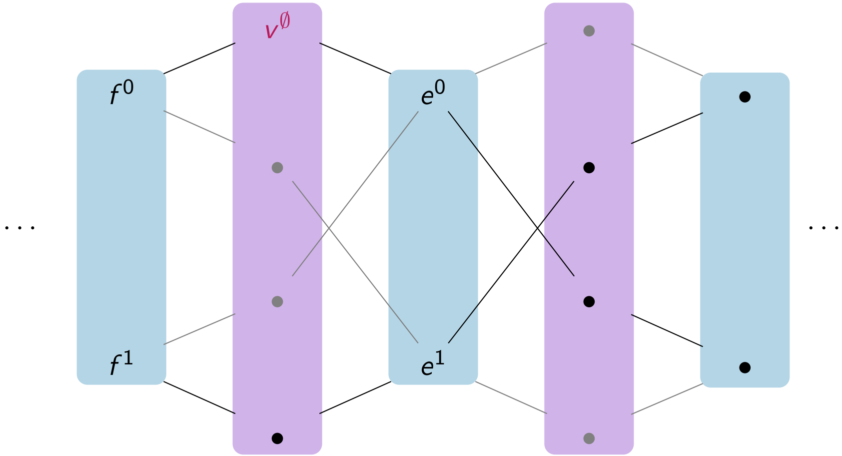

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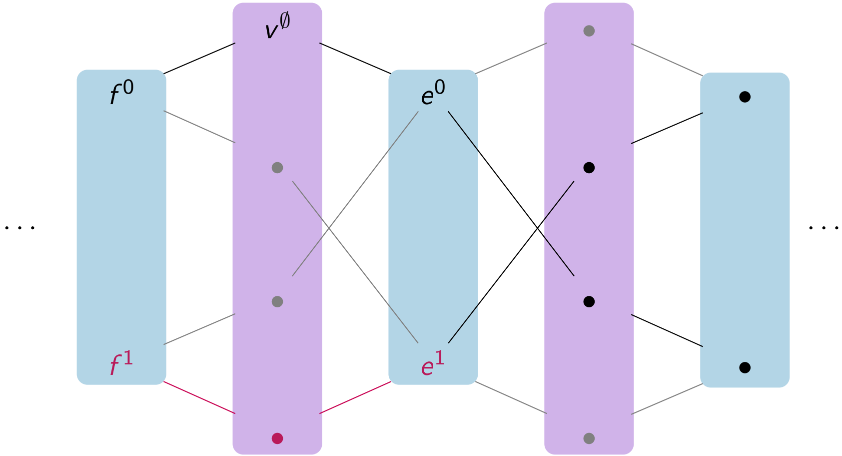

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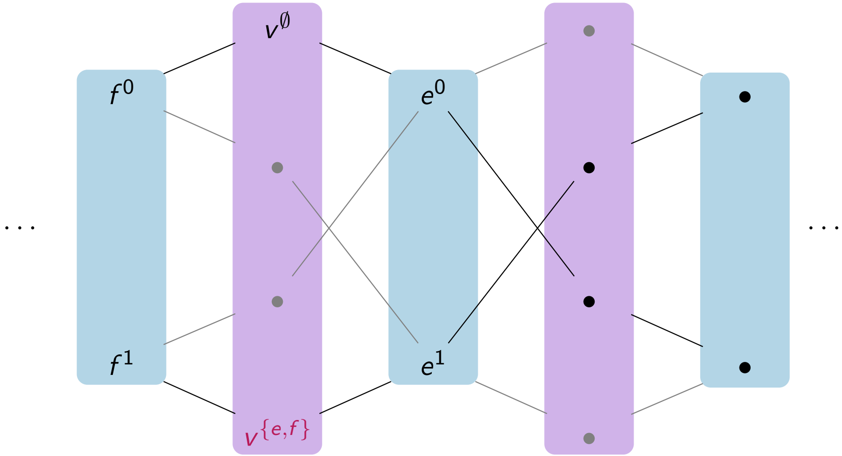

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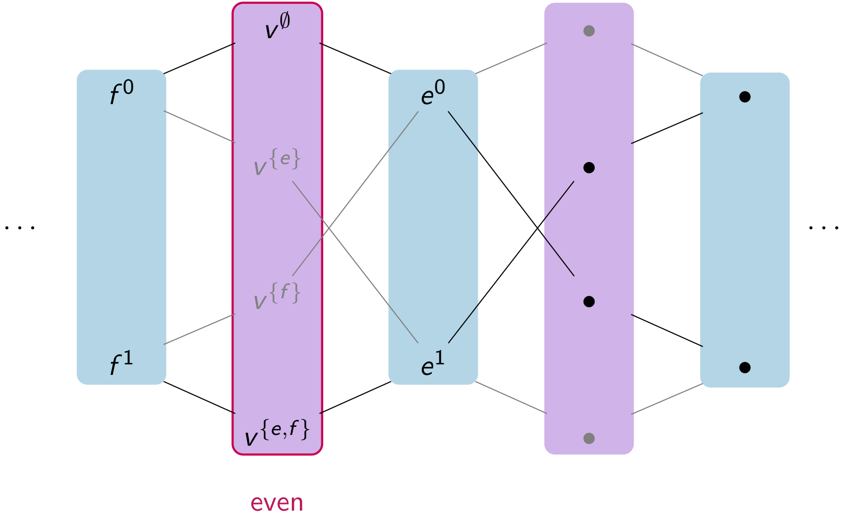

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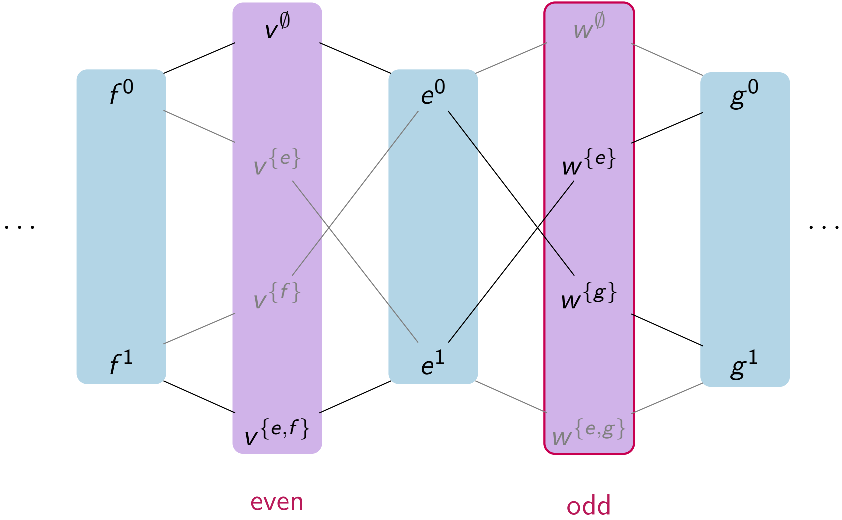

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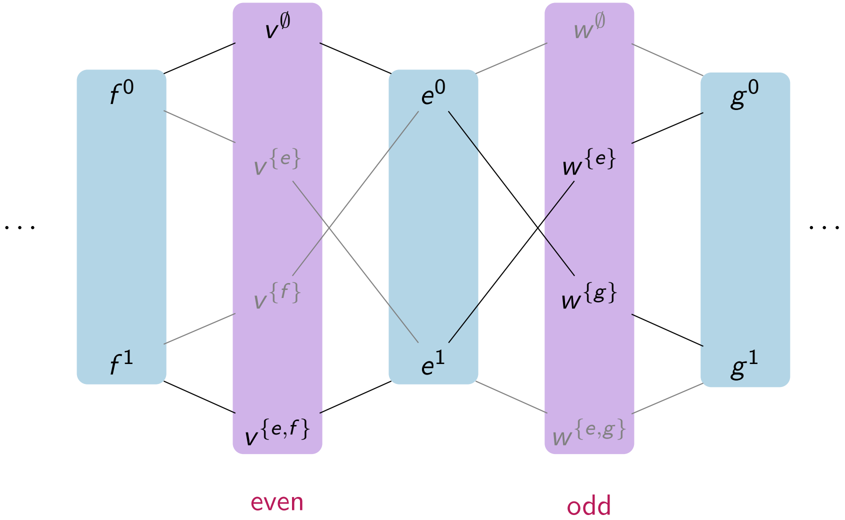
CPT procedure (Dawar, Richerby, Rossman)

1. Construct super-symmetric objects
2. Label edge gadgets
3. Count odd vertex gadgets

Why super-symmetry is useful



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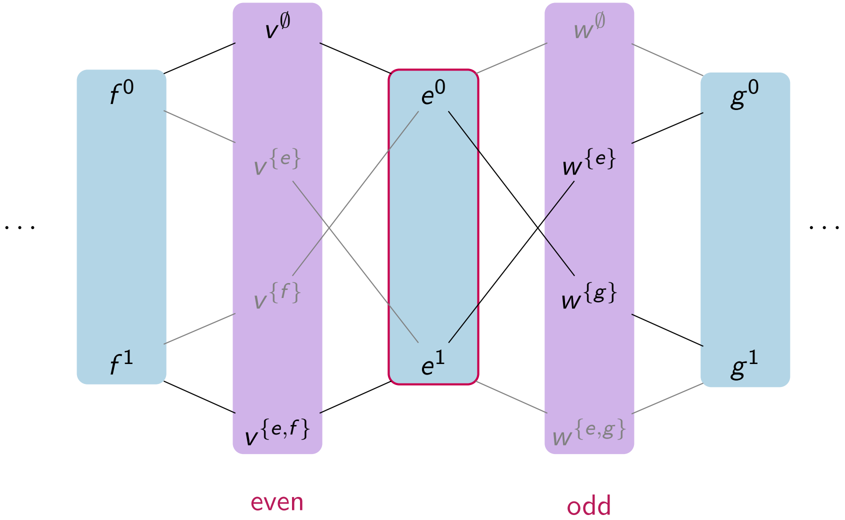

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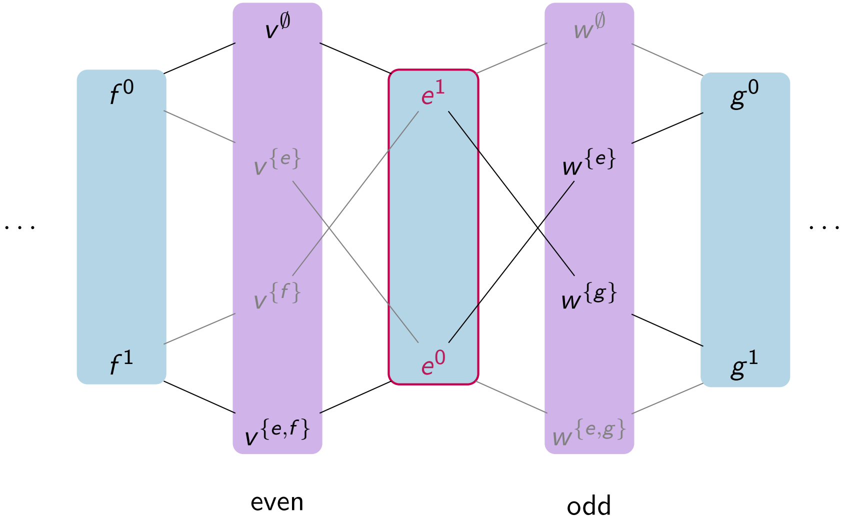

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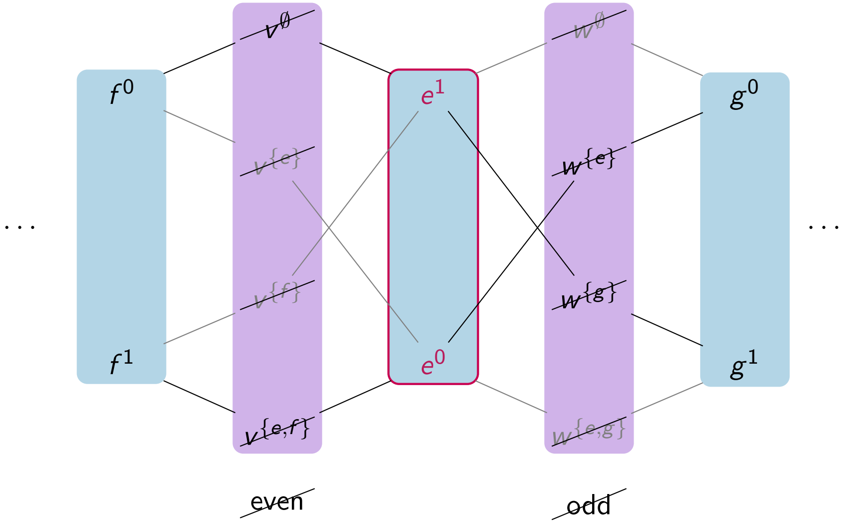

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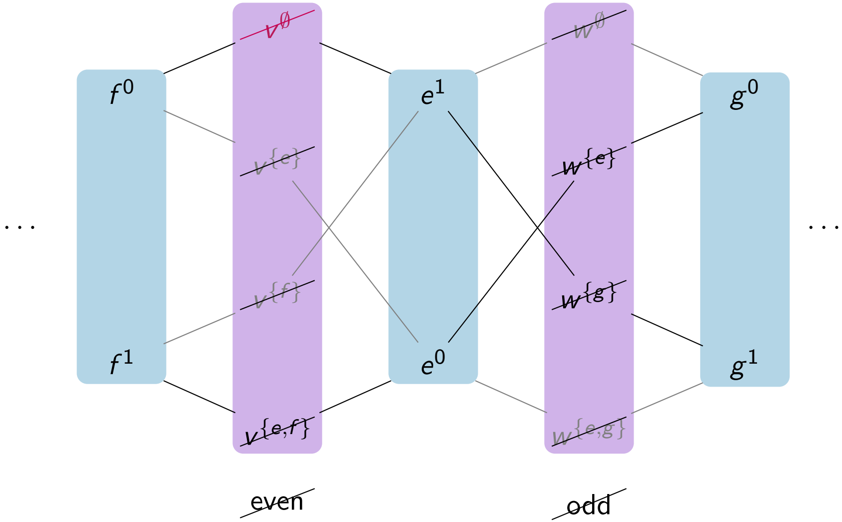

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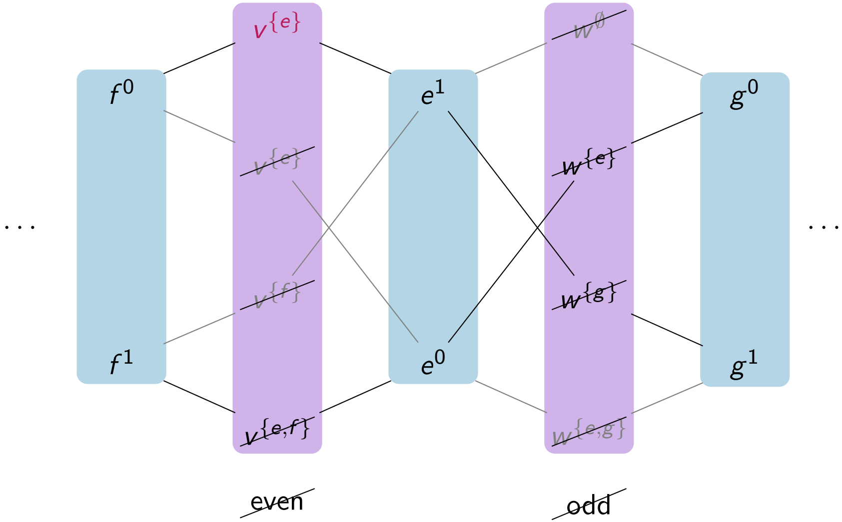

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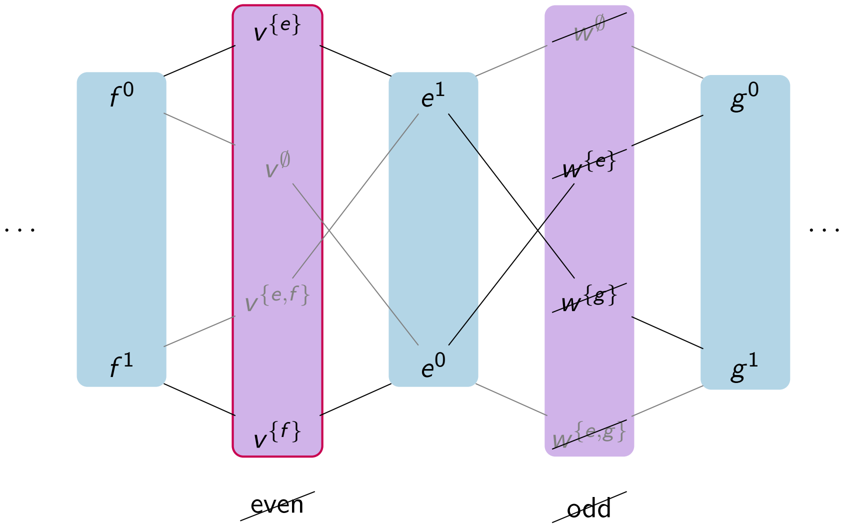

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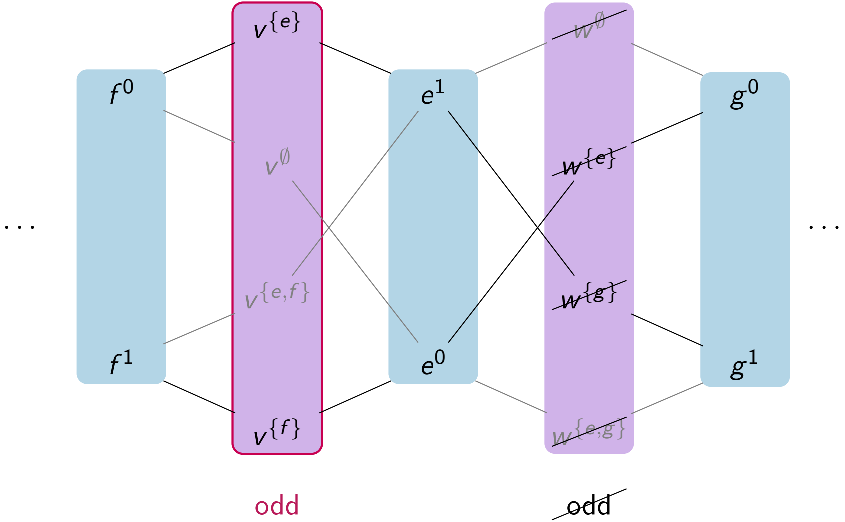

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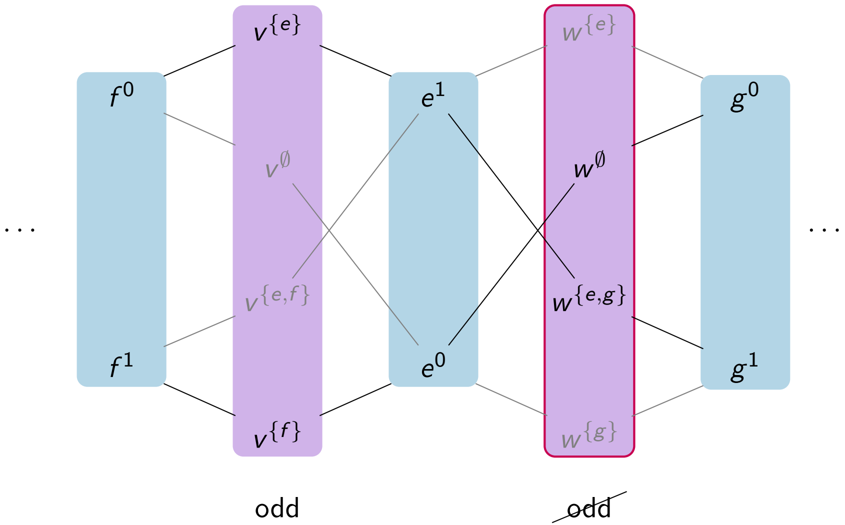

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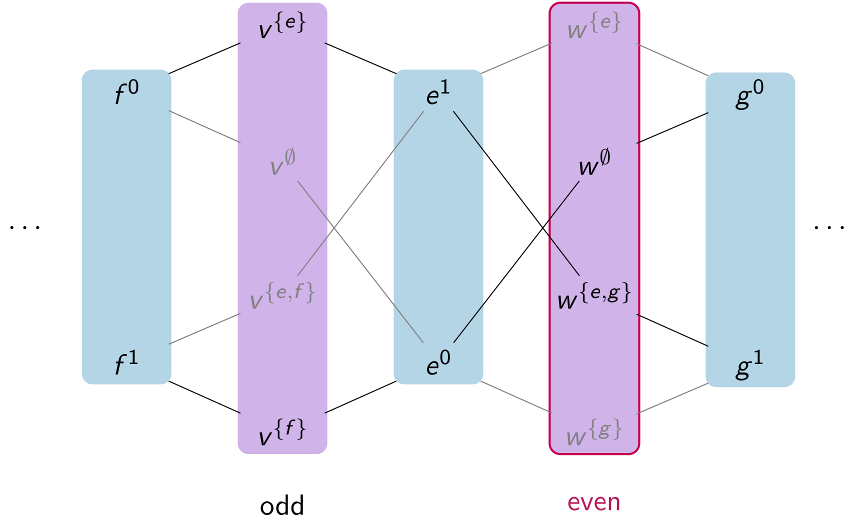

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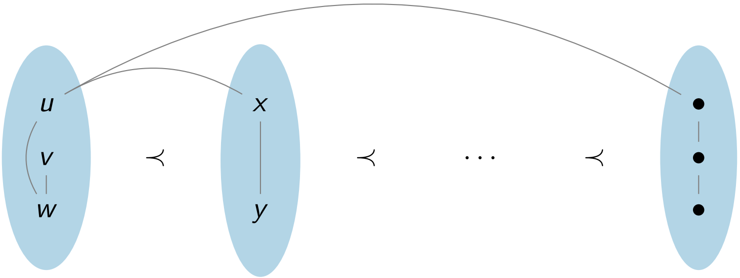
Graphs with colour classes of logarithmic size



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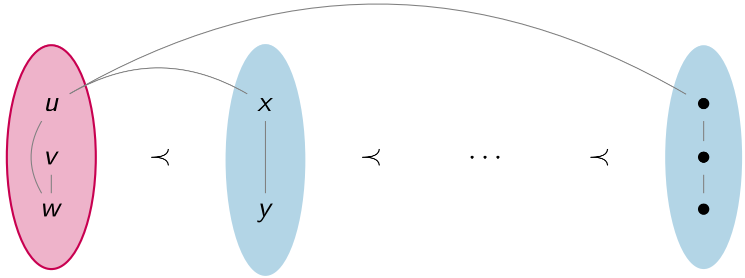

$\mu_{\{u\}}$ $\mu_{\{v\}}$ $\mu_{\{w\}}$

$\mu_{\{u,v\}}$ $\mu_{\{u,w\}}$ $\mu_{\{v,w\}}$

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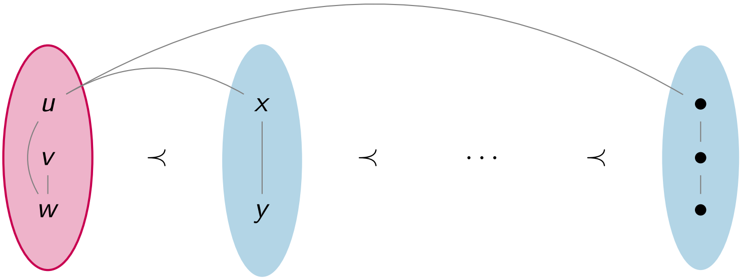
Graphs with colour classes of logarithmic size



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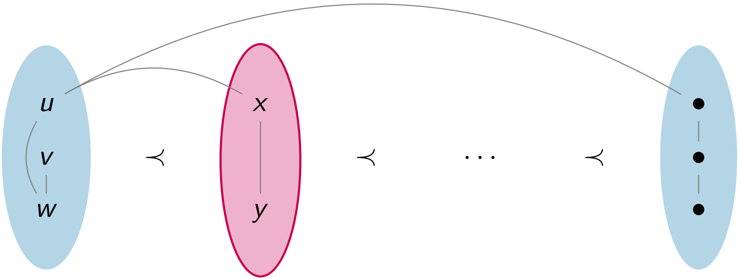
Construct $\mathcal{O}(2^{|C|})$ many sets

Graphs with colour classes of logarithmic size

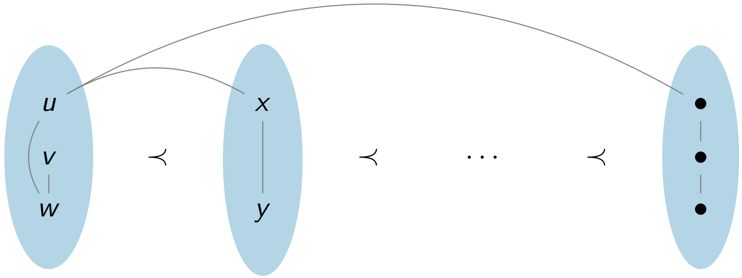


$\mu_{\{u,v,w\}}$

$\mu_{\{u,v,w,x\}}$ $\mu_{\{u,v,w,y\}}$

$\mu_{\{u,v,w,x,y\}}$

Graphs with colour classes of logarithmic size



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μ_V

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Theorem (Dawar, Richerby, Rossman)

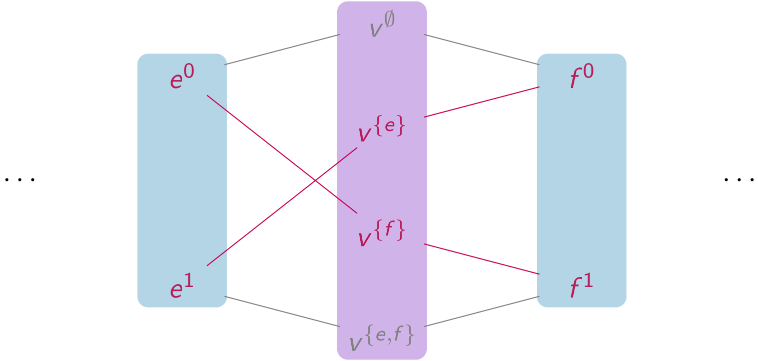
The CFI-query over ordered graphs is not definable in CPT using only sets of bounded rank.

Graphs with large degree: Keeping the rank small

- Access to all subsets of V

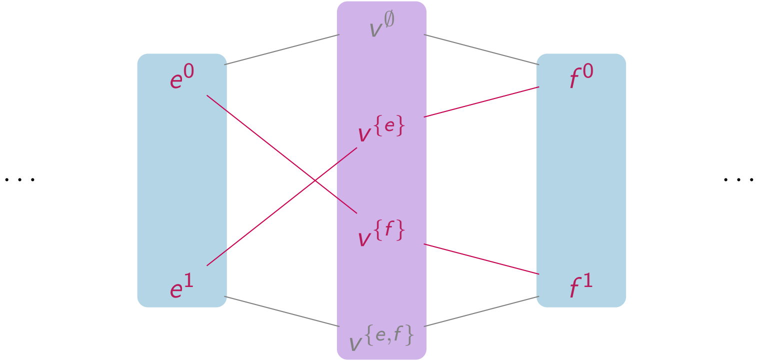
Graphs with large degree: Keeping the rank small

- Access to all subsets of V



Graphs with large degree: Keeping the rank small

- Access to all subsets of V
- Intuition: “Ordered” objects need nesting



Theorem
The CFI query over graphs with logarithmic colour classes is CPT*-definable.* ✓

Theorem
The CFI query over graphs with large degree is CPT*-definable using only sets of bounded rank.* ✓

Theorem
The CFI query over complete graphs is not CPT*-definable without using set-like objects.*

Corollary
$\approx$ *-free* $\mathrm{PIL} \not\equiv \mathrm{CPT}[\mathrm{rk} \leq k]$

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Proving lower bounds

CPT-program $\rightsquigarrow$ $\mathcal{C}^k$-formula over HF($\mathfrak{A}$)

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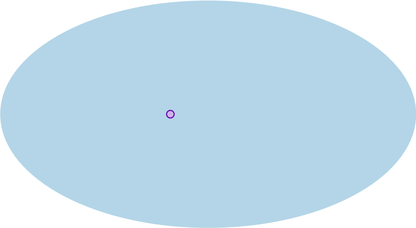
CPT-program $\rightsquigarrow$ $\mathcal{C}^k$-formula over $\mathrm{HF}(\mathfrak{A})$

Move in $\mathrm{HF}(\mathfrak{A})$:



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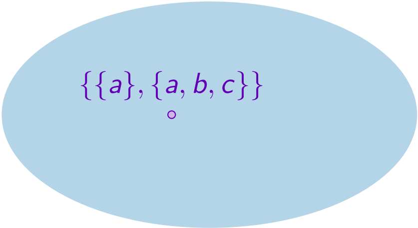
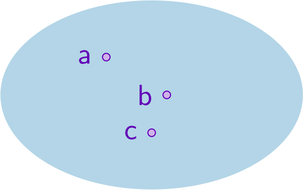


CPT-program $\rightsquigarrow \mathcal{C}^k$-formula over HF($\mathfrak{A}$)

"Corresponding" move in $\mathfrak{A}$:



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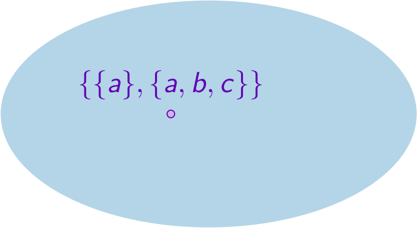

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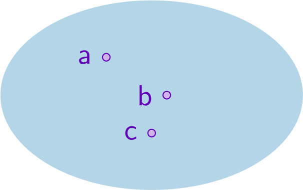

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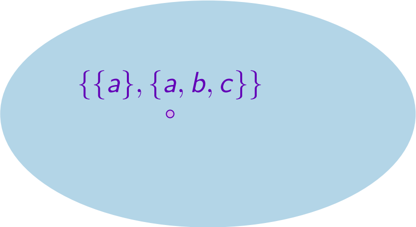

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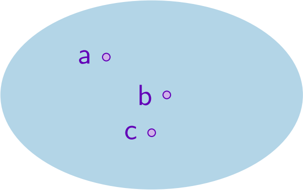

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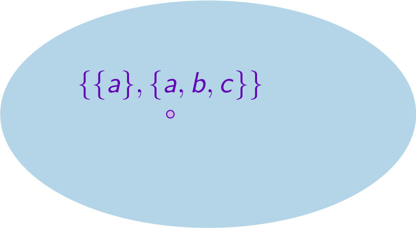

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Example

$A = \{a, b, c, d, e\}$

Supports of $\{a, b, c\}$: A, $\{a, b, c\}$, $\{d, e\}$

Proving lower bounds

sets with small support

CPT-program $\rightsquigarrow$ $\mathcal{C}^k$-formula over $\mathrm{HF}(\mathfrak{A})_\ell$

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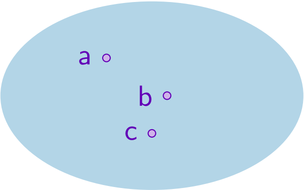

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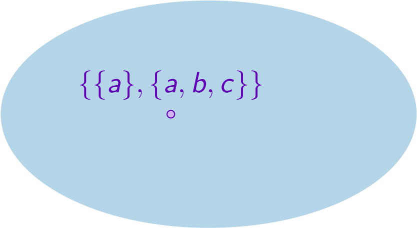

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Sequence-like objects: strong supports

$v_1, \ldots, v_k$ vertices of $\mathcal{K}_n$.

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$y = (v_1, \ldots, v_k)$

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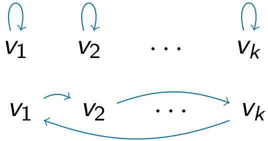

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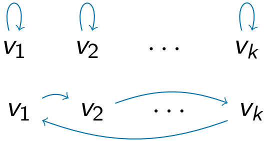

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$\rightsquigarrow$ strong support

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