Model Checking for Invariant FO on Restricted Graph Classes

Kord Eickmeyer, TU Darmstadt
joint work with Ken-ichi Kawarabayashi, NII Tokyo

CSL, August 30th, 2016

The Main Results

Theorem (Main Result I)

For every $r \in \mathbb{N}$, the model-checking problem for successor-invariant first-order logic (succ-inv-FO) *on the class of all finite structures whose Gaifman graph does not contain K_r as a topological subgraph is fixed-parameter tractable when parameterised by the size of the formula and r.*

Theorem (Main Result II)

For every $w \in \mathbb{N}$, the model-checking problem for order-invariant first-order logic (<-inv-FO) *on the class of coloured posets of width $\leq w$ is fixed-parameter tractable when parameterised by the size of the formula and w.*

succ-inv FO on Graphs Excluding a Topological Subgraph

Theorem (Main Result I)

For every $r \in \mathbb{N}$, the model-checking problem for successor-invariant first-order logic (succ-inv-FO) on the class of all finite structures whose Gaifman graph does not contain K_r as a topological subgraph is fixed-parameter tractable when parameterised by the size of the formula and r.

- extends previous results by Engelmann, Kreutzer, Siebertz (planar graphs, LICS 2012) and by E, Kawarabayashi, Kreutzer (excluded minors, LICS 2013)
- stronger results for plain FO are known

Logics with Invariant Relations

- (all structures are finite, relational)
- $<$-inv-FO: Allow invariant use of order an relation
- formally: for signature σ such that $< \notin \sigma$, consider all FO$[\sigma \cup \{<\}]$-sentences φ such that

$$A, <_1 \models \varphi \qquad \text{iff} \qquad A, <_2 \models \varphi$$

  for all finite σ-structures A and all orders $<_1$, $<_2$ on A.
- similar: succ-inv-FO, $+$-inv-FO, ...
- semantic definition, not closed under subformulae, no EF-games, no Feferman-Vaught, no Gaifman normal form, ...

Facts about $<$-inv-FO and succ-inv-FO

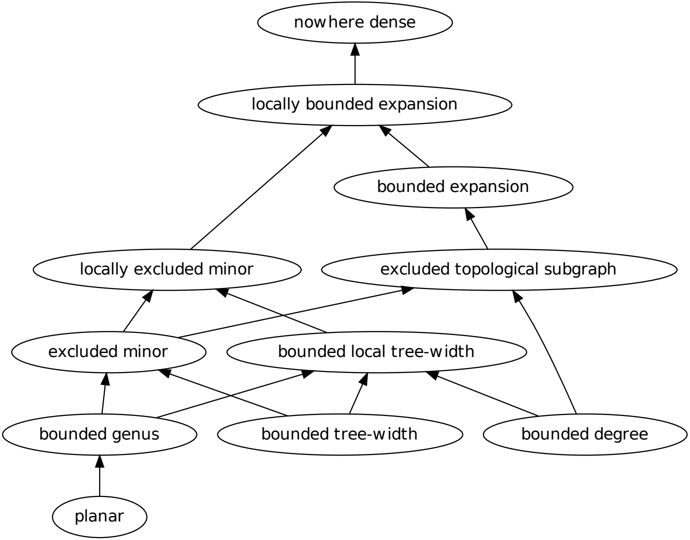
- natural extensions of FO in the context of model-checking (representations of structures $\rightsquigarrow$ linear order)
- $<$-inv-FO is stronger than FO on finite structures (Gurevich, Potthoff, Otto): Boolean algebras with an even number of atoms
- succ-inv-FO is stronger than FO on finite structures (Rossman)
- $<$-inv-FO $\equiv$ FO on various kinds of finite trees (Benedict and Segoufin)
- $<$-inv-FO $\equiv$ FO on siblinged unranked trees of bounded height (Dawar and E, unpublished)
- $<$-inv-MSO $\equiv$ FO+MOD on structures of bounded tree-depth (E, Elberfeld, Harwath, MFCS 2014)

- basic problem: Given formula φ and structure A, is $A \models \varphi$?
- for $\varphi \in$ FO PSPACE-complete even for $A = \{0,1\}$
- AW[*]-complete when parameterised by $|\varphi|$
- fixed-parameter tractable if A is restricted to structures with nice properties
- fixed-parameter tractable means: running time

$$f(\varphi) \cdot |A|^c$$

for some *fixed* c (independent of φ).

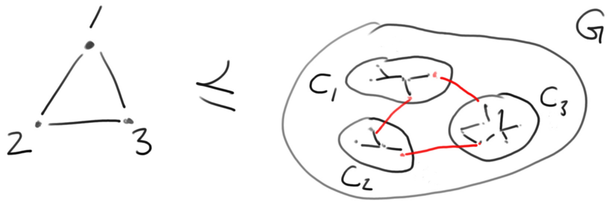
(plain) FO model checking is fpt on...



Minors

- $G = (V, E)$, $H = (W, F)$ graphs
- H is a minor of G (written $H \preceq G$) if there are connected, vertex-disjoint subgraphs $C_u \leq G$ for each $u \in W$ such that

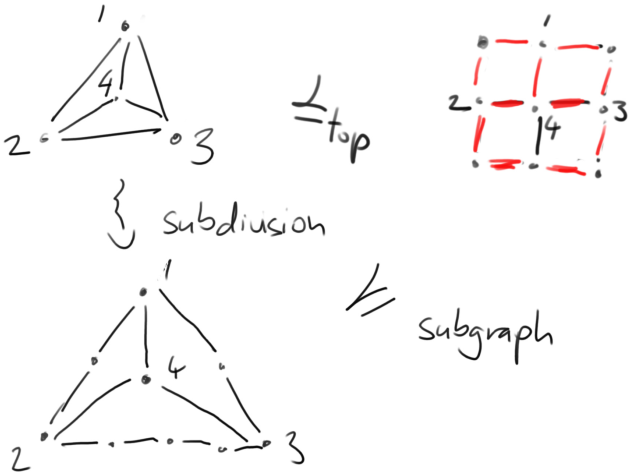
$$uv \in F \quad \Rightarrow \quad \text{there are } x \in C_u, y \in C_v \text{ such that } xy \in E.$$



- equivalent: can obtain H from G by vertex/edge deletion and edge contraction

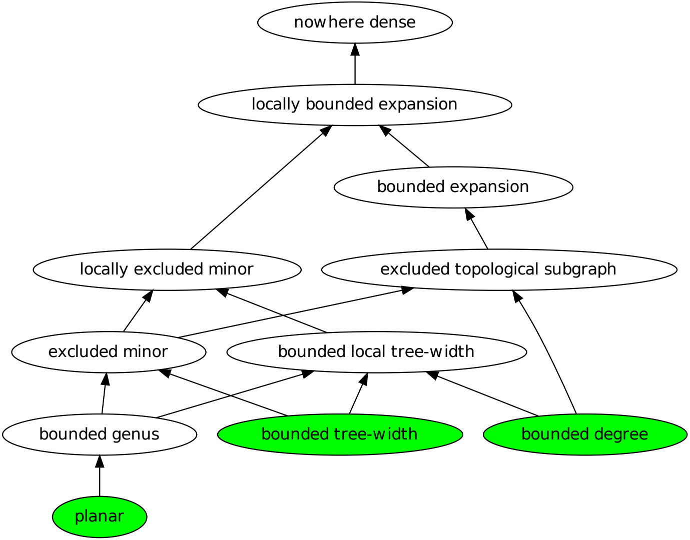
Topological Subgraphs

- H is a topological subgraph of G (written $H \preceq_{\text{top}} G$) if some subdivision of H is a subgraph of G



What about succ-inv-FO?

- recall: semantic definition, not closed under subformulae, no EF-games, no Feferman-Vaught, no Gaifman normal form, ...
- for succ-inv-FO: successor relations $\leftrightarrow$ Hamiltonian paths
- add Hamiltonian path in a clever way, so that good graphs remain good!


nowhere dense
locally bounded expansion
bounded expansion
locally excluded minor
excluded topological subgraph
excluded minor
bounded local tree-width
bounded genus
bounded tree-width
bounded degree
planar

Relaxation to k-walks

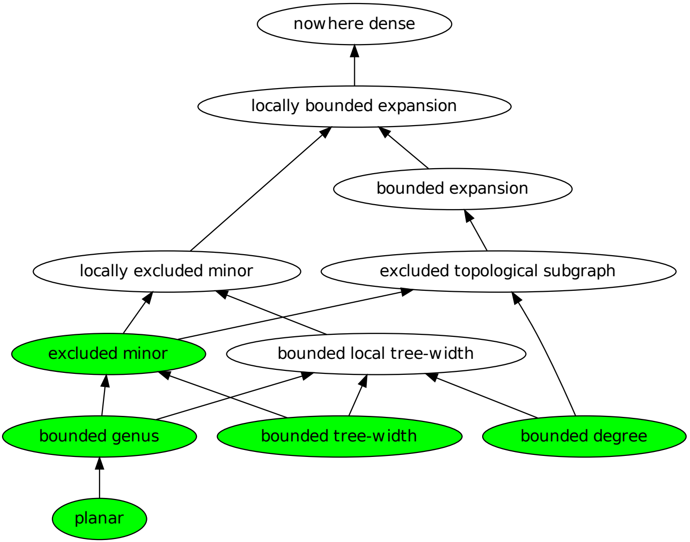
- using an interpretation argument, we relax our requirement from Hamiltonian path to k-walk
- k-walk through G: visit each vertex at least once, at most k-times

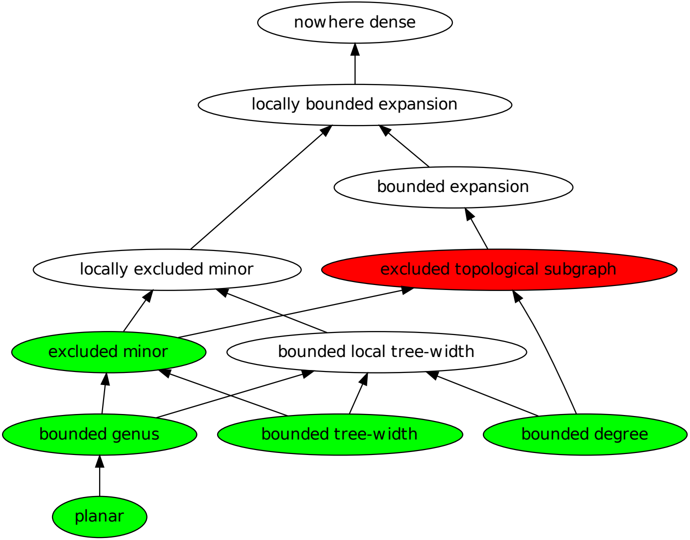
from k-walks to Hamiltonian paths

Let σ be a finite relational vocabulary, A a finite σ-structure, and $w : [\ell] \to V(A)$ a k-walk through the Gaifman graph of A.
Then there is a finite relational vocabulary σ_k and a first-order fomula $\varphi^{(k)}_{\mathrm{succ}}(x, y)$, both depending only on k, and a $(\sigma \cup \sigma_k)$-expansion A' of A which can be computed from A and w in polynomial time, such that

- The Gaifman-graphs of A' and A are the same,
- $\varphi^{(k)}_{\mathrm{succ}}$ defines a successor relation on A'.

succ-inv-FO: E, Kawarabayashi, Kreutzer (LICS2013)




nowhere dense
locally bounded expansion
bounded expansion
locally excluded minor
excluded topological subgraph
excluded minor
bounded local tree-width
bounded genus
bounded tree-width
bounded degree
planar

Decomposing Graphs with Excluded Topological Subgraphs

We apply the following result by Grohe and Marx (2015):

Structure Thm for Excluded Topological Subgraphs

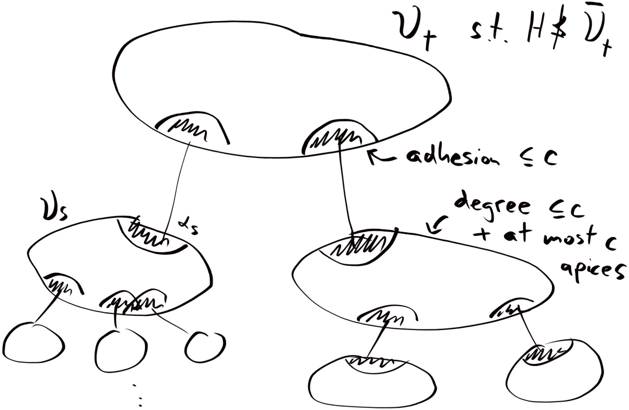
For every $k \in \mathbb{N}$ there exists a constant $c = c(k) \in \mathbb{N}$ such that the following holds: If H is a a graph on k vertices and G a graph which does not contain H as a topological subgraph, then there is a tree-decomposition $(\mathcal{T}, \mathcal{V})$ of G of adhesion at most c such that for all $t \in \mathcal{T}$

- $\bar{\mathcal{V}}_t$ has at most c vertices of degree larger than c, *or*
- $\bar{\mathcal{V}}_t$ excludes K_c as a minor.

Furthermore, there is an algorithm that, given graphs G of size n and H of size k computes such a decomposition in time $f(k) \cdot n^{O(1)}$ for some computable function $f : \mathbb{N} \to \mathbb{N}$.

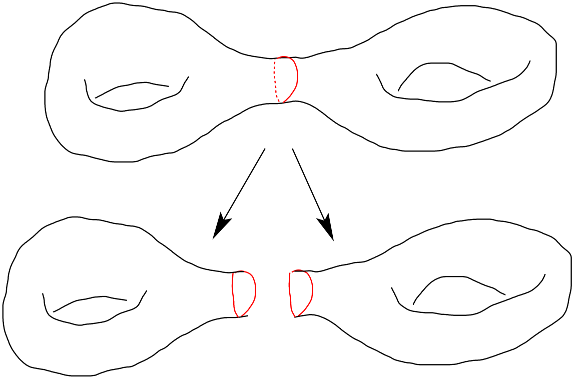
Grohe/Marx Theorem

- decompose input graph into H-minor free bags and bags of almost bounded degree.



k-walks in Graphs with Excluded Minors

- then compute k-walks in $\mathcal{V}_t \setminus \alpha_t$ for all nodes t of the tree
- for H-minor free bags: use graph structure theorem, induction on genus
- 2-walks exist in 3-connected planar graphs (Gao and Richter)
- need to add edges within $\mathcal{V}_t$, but still H'-minor free

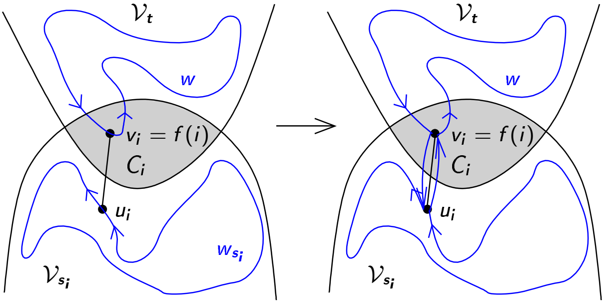


k-walks in Graphs with bounded degree

- for bags with almost bounded degree: choose an arbitrary Hamiltonian path
- increases degrees by at most 2

Putting the k-walks together

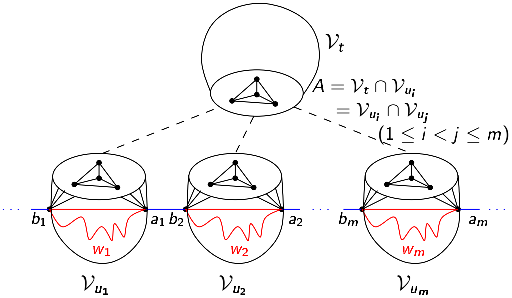
- then, connect these k-walks to obtain one k-walk through the whole graph



- degree of connecting vertex v_i increases by one
- must choose v_i so that no $v \in V$ is used more than $O(1)$ times!

Putting the k-walks together (contd.)

- if there are siblings with identical adhesion, insert new edges



- otherwise, let $C_1, \ldots, C_k$ be adhesion sets which are not yet connected
- then $G[\bigcup_i C_i]$ has some vertex of small degree (note: excluded minor $\Rightarrow$ degenerate)
- vertices of small degree can only be in few adhesion sets (since those are cliques)
- succesively connect along vertices of small degree

$<$-inv FO on Posets of Bounded Width

Theorem (Main Result II)

For every $w \in \mathbb{N}$, the model-checking problem for order-invariant first-order logic ($<$-inv-FO) on the class of coloured posets of width $\leq w$ is fixed-parameter tractable when parameterised by the size of the formula and w.

- For FO: Recent result by Gajarský et al.
- width: maximum length of antichain
- Dilworth's Thm: width $k \rightsquigarrow$ union of k chains

The remaining gap for succ-inv-FO, results for <-inv-FO?

- Conclusion: We showed that succ-inv-FO is tractable on graphs excluding a topological subgraph and <-inv-FO is tractable on bounded width posets.

- Still open: locally excluded minors, bounded local tree-width, (locally) bounded expansion, nowhere dense
- no strong structure theorems for these graph classes

- what about <-inv-FO?
- problem: Gaifman graph of linear order is a clique
- locality of <-inv-FO is known, but no Gaifman normal form