

Model Checking for Invariant FO on Restricted Graph Classes

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The Main Results

Theorem (Main Result I)

For every $r \in \mathbb{N}$, the model-checking problem for successor-invariant first-order logic (succ-inv-FO) on the class of all finite structures whose Gaifman graph does not contain K_r as a topological subgraph is fixed-parameter tractable when parameterised by the size of the formula and r .

Theorem (Main Result II)

For every $w \in \mathbb{N}$, the model-checking problem for order-invariant first-order logic (<-inv-FO) on the class of coloured posets of width $\leq w$ is fixed-parameter tractable when parameterised by the size of the formula and w .

Theorem (Main Result I)

For every $r \in \mathbb{N}$, the model-checking problem for successor-invariant first-order logic (succ-inv-FO) on the class of all finite structures whose Gaifman graph does not contain K_r as a topological subgraph is fixed-parameter tractable when parameterised by the size of the formula and r .

- extends previous results by Engelmann, Kreutzer, Siebertz (planar graphs, LICS 2012) and by E, Kawarabayashi, Kreutzer (excluded minors, LICS 2013)
- stronger results for plain FO are known

Logics with Invariant Relations

- (all structures are finite, relational)
- $<$ -inv-FO: Allow invariant use of order an relation
- formally: for signature σ such that $< \notin \sigma$, consider all $\text{FO}[\sigma \cup \{<\}]$ -sentences φ such that

$$A, <_1 \models \varphi \quad \text{iff} \quad A, <_2 \models \varphi$$

for all finite σ -structures A and all orders $<_1, <_2$ on A .

- similar: succ-inv-FO, +-inv-FO, ...
- semantic definition, not closed under subformulae, no EF-games, no Feferman-Vaught, no Gaifman normal form, ...

Facts about $<-inv-FO$ and $succ-inv-FO$

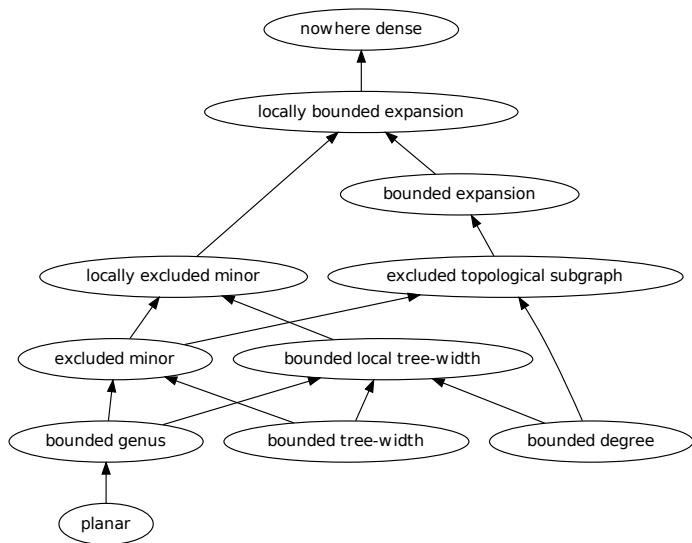
- natural extensions of FO in the context of model-checking (representations of structures \rightsquigarrow linear order)
- $<-inv-FO$ is stronger than FO on finite structures (Gurevich, Potthoff, Otto): Boolean algebras with an even number of atoms
- $succ-inv-FO$ is stronger than FO on finite structures (Rossman)
- $<-inv-FO \equiv FO$ on various kinds of finite trees (Benedict and Segoufin)
- $<-inv-FO \equiv FO$ on siblinged unranked trees of bounded height (Dawar and E, unpublished)
- $<-inv-MSO \equiv FO+MOD$ on structures of bounded tree-depth (E, Elberfeld, Harwath, MFCS 2014)

- basic problem: Given formula φ and structure A , is $A \models \varphi$?
- for $\varphi \in \text{FO}$ PSPACE-complete even for $A = \{0, 1\}$
- AW[*]-complete when parameterised by $|\varphi|$
- fixed-parameter tractable if A is restricted to structures with nice properties
- fixed-parameter tractable means: running time

$$f(\varphi) \cdot |A|^c$$

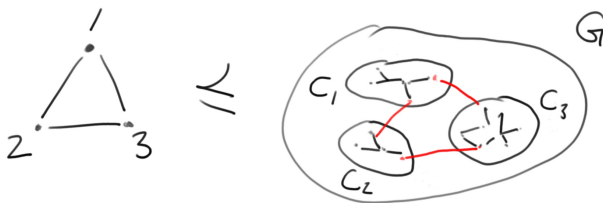
for some *fixed* c (independent of φ).

(plain) FO model checking is fpt on...



Minors

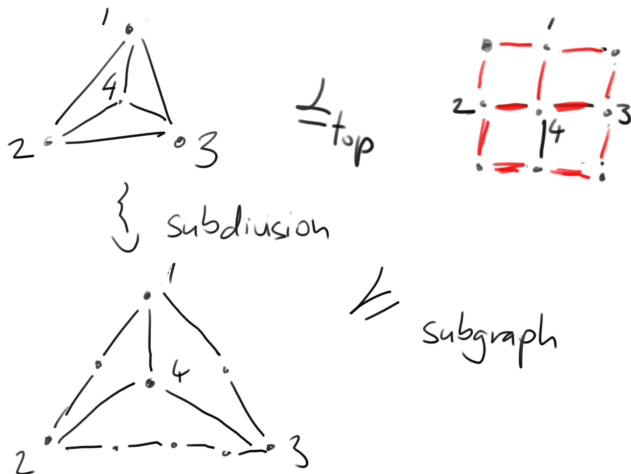
- $G = (V, E)$, $H = (W, F)$ graphs
- H is a minor of G (written $H \preceq G$) if there are connected, vertex-disjoint subgraphs $C_u \subseteq G$ for each $u \in W$ such that
$$uv \in F \Rightarrow \text{there are } x \in C_u, y \in C_v \text{ such that } xy \in E.$$



- equivalent: can obtain H from G by vertex/edge deletion and edge contraction

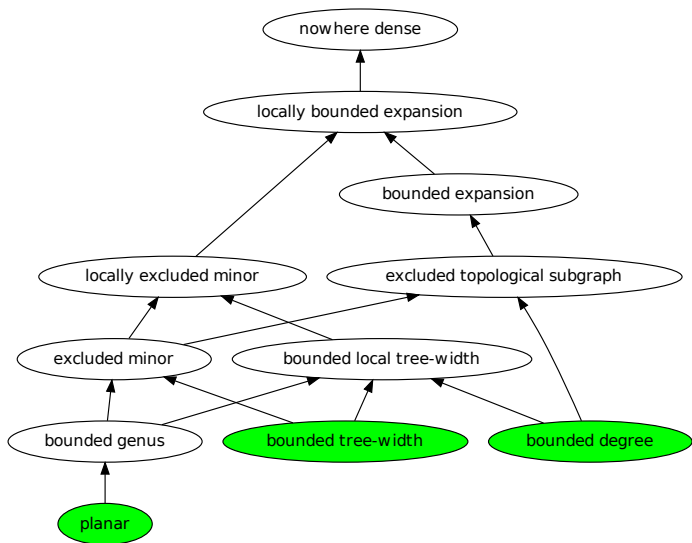
Topological Subgraphs

- H is a topological subgraph of G (written $H \preceq_{\text{top}} G$) if some subdivision of H is a subgraph of G



What about succ-inv-FO ?

- recall: semantic definition, not closed under subformulae, no EF-games, no Feferman-Vaught, no Gaifman normal form, ...
- for succ-inv-FO : successor relations \leftrightarrow Hamiltonian paths
- add Hamiltonian path in a clever way, so that good graphs remain good!



Relaxation to k -walks

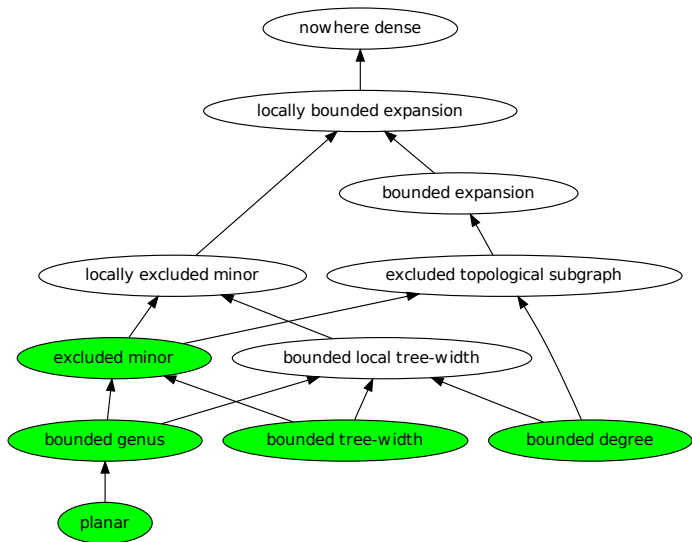
- using an interpretation argument, we relax our requirement from Hamiltonian path to k -walk
- k -walk through G : visit each vertex at least once, at most k -times

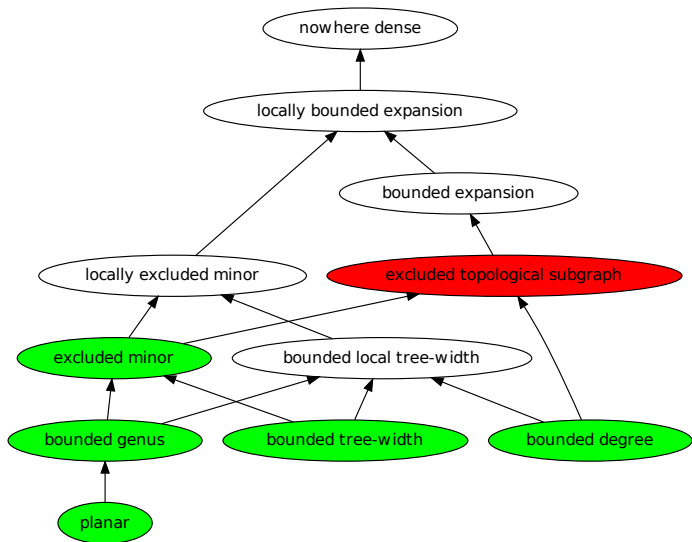
from k -walks to Hamiltonian paths

Let σ be a finite relational vocabulary, A a finite σ -structure, and $w : [\ell] \rightarrow V(A)$ a k -walk through the Gaifman graph of A .

Then there is a finite relational vocabulary σ_k and a first-order formula $\varphi_{\text{succ}}^{(k)}(x, y)$, both depending only on k , and a $(\sigma \cup \sigma_k)$ -expansion A' of A which can be computed from A and w in polynomial time, such that

- The Gaifman-graphs of A' and A are the same,
- $\varphi_{\text{succ}}^{(k)}$ defines a successor relation on A' .





We apply the following result by Grohe and Marx (2015):

Structure Thm for Excluded Topological Subgraphs

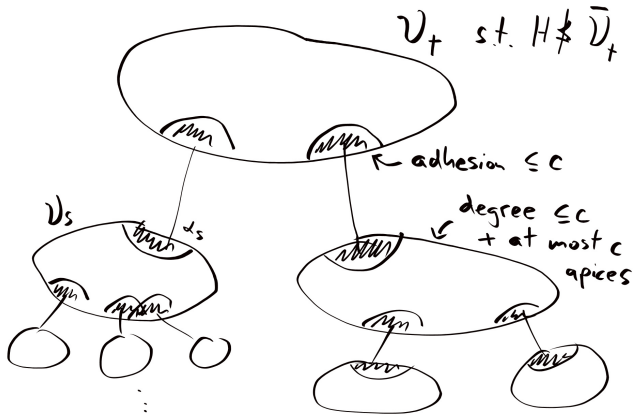
For every $k \in \mathbb{N}$ there exists a constant $c = c(k) \in \mathbb{N}$ such that the following holds: If H is a graph on k vertices and G a graph which does not contain H as a topological subgraph, then there is a tree-decomposition $(\mathcal{T}, \mathcal{V})$ of G of adhesion at most c such that for all $t \in \mathcal{T}$

- $\bar{\mathcal{V}}_t$ has at most c vertices of degree larger than c , or
- $\bar{\mathcal{V}}_t$ excludes K_c as a minor.

Furthermore, there is an algorithm that, given graphs G of size n and H of size k computes such a decomposition in time $f(k) \cdot n^{O(1)}$ for some computable function $f : \mathbb{N} \rightarrow \mathbb{N}$.

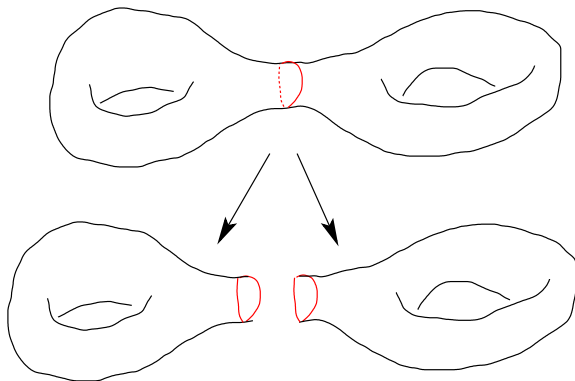
Grohe/Marx Theorem

- decompose input graph into H -minor free bags and bags of almost bounded degree.



k -walks in Graphs with Excluded Minors

- then compute k -walks in $\mathcal{V}_t \setminus \alpha_t$ for all nodes t of the tree
- for H -minor free bags: use graph structure theorem, induction on genus
- 2-walks exist in 3-connected planar graphs (Gao and Richter)
- need to add edges within \mathcal{V}_t , but still H' -minor free

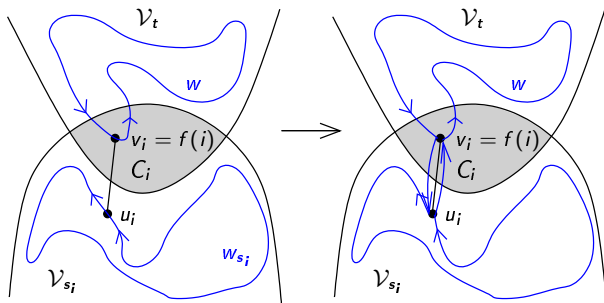


k -walks in Graphs with bounded degree

- for bags with almost bounded degree: choose an arbitrary Hamiltonian path
- increases degrees by at most 2

Putting the k -walks together

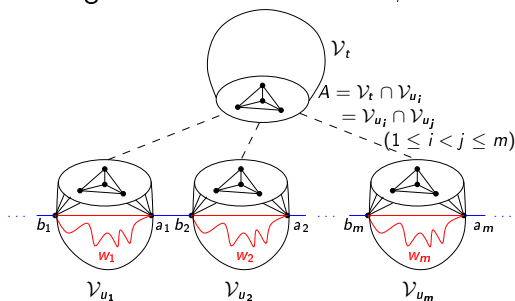
- then, connect these k -walks to obtain one k -walk through the whole graph



- degree of connecting vertex v_i increases by one
- must choose v_i so that no $v \in V$ is used more than $O(1)$ times!

Putting the k -walks together (contd.)

- if there are siblings with identical adhesion, insert new edges



- otherwise, let C_1, \dots, C_k be adhesion sets which are not yet connected
- then $G[\bigcup_i C_i]$ has some vertex of small degree (note: excluded minor \Rightarrow degenerate)
- vertices of small degree can only be in few adhesion sets (since those are cliques)
- successively connect along vertices of small degree

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- For FO: Recent result by Gajarský et al.
- width: maximum length of antichain
- Dilworth's Thm: width $k \rightsquigarrow$ union of k chains

The remaining gap for succ-inv-FO , results for $\prec\text{-inv-FO}$?

- Conclusion: We showed that succ-inv-FO is tractable on graphs excluding a topological subgraph and $\prec\text{-inv-FO}$ is tractable on bounded width posets.
- Still open: locally excluded minors, bounded local tree-width, (locally) bounded expansion, nowhere dense
- no strong structure theorems for these graph classes
- what about $\prec\text{-inv-FO}$?
- problem: Gaifman graph of linear order is a clique
- locality of $\prec\text{-inv-FO}$ is known, but no Gaifman normal form