# Context-Free Graph Properties via Definable Decompositions


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# MSO-logic is used outside model theory.

## Monadic second-order (MSO) logic

**Syntax** $\varphi$ quantifies sets $\exists X \subseteq U$ besides $\exists x \in U$,

**Semantics** defines property $\{\text{finite structure } \mathcal{A} \mid \mathcal{A} \models \varphi\}$.

## Strings of even length are MSO-definable

**Language** $L = \{\mathrm{a}^n \mid n \text{ is even}\}$ is MSO-definable since

**Property** $\textsc{structure}(L) = \{\, \circ\!\to\!\circ,\ \circ\!\to\!\circ\!\to\!\circ\!\to\!\circ\,,\ \ldots\,\}$ is MSO-definable via guess-and-check coloring.

## Language-Theoretic Fact

*Regular languages are* MSO*-definable, and vice versa.*

[Büchi, 1960, Elgot, 1961, Trakhtenbrot, 1961]

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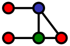
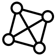
## 3-colorable graphs are MSO-definable

We have  $\models \varphi$, but  $\not\models \varphi$ with MSO-formula

$$\begin{aligned}\varphi = \exists\, R, G, B \subseteq U\; & \forall\, v \in U \left[\, v \in R \vee v \in G \vee v \in B \,\right] \wedge \\ & \forall\, (v, w) \in E\;\; \neg\left[\, v, w \in R \vee v, w \in G \vee v, w \in B \,\right]\end{aligned}$$

## Algorithmic Fact

*Each* MSO*-definable property is solvable in linear time on structures with bounded tree width.* [Courcelle, 1990]

# Second-order logic is too general.

## Second-order (SO) logic

Syntax $\varphi$ uses higher-ary quantifiers $\exists X \subseteq U \times U$,
Semantics defines property $\{\text{finite structure } \mathcal{A} \mid \mathcal{A} \models \varphi\}$.

## Complexity Fact

*The SO-definable properties of strings are the problems of the polynomial hierarchy (PH).* [Stockmeyer, 1976]

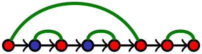
## SO's expressivity and complexity

Since P $\subseteq$ NP $\subseteq$ PH $\subseteq$ PSPACE,

- SO-definable properties are far beyond regular and context-free, and
- modest polynomial runtimes are out of reach.

# A matching relation contains nested tuples.

## String structures with matchings

Structure $(\mathcal{S}, \mathcal{M}) =$  representing $s =$ 10101111 and well-formed $m =$ (() ()) ().

## The logic "$\exists$ Match MSO" for string structures

**Syntax** $\psi = \exists^{\text{matching}} M\, [\varphi]$ with MSO-formula $\varphi$.

**Semantics** String structure $\mathcal{S} \models \psi$ if $(\mathcal{S}, \mathcal{M}) \models \varphi$ for a matching $\mathcal{M}$ of $\mathcal{S}$.

## Language-Theoretic Fact

*The "$\exists$ Match MSO"-definable properties of strings are the context-free languages.*

[Lautemann, Schwentick, and Thérien, 1995]

# Paper generalizes from strings to graphs.

## Research question

"a more general study of these logics will prove worthwhile in the context of general finite structures, instead of strings."

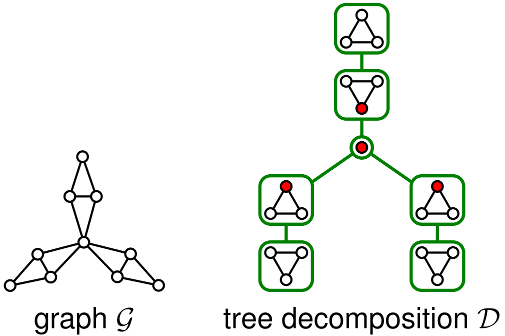
[Lautemann, Schwentick, and Thérien, 1995]

## Paper's contribution

A logic for structures with bounded tree width along with understanding the following aspects:

- **languages** Its equivalence to context-free languages.
- **model theory** Its expressive power.
- **algorithms** Its satisfiability and evaluation problems.

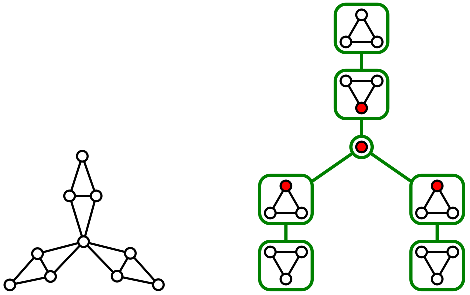
# We use decompositions of bounded width.




## Properties of our tree decompositions

| | |
|---|---|
| Connected | For every vertex, its bags are connected. |
| Covering | Bags cover all edges |
| Strictness | Subtrees cover ever “smaller” subgraphs. |
| Normal | Subtrees cover connected subgraphs. |

# We use decompositions of bounded width.



graph $\mathcal{G}$ tree decomposition $\mathcal{D}$

## Width of tree decompositions

width($\mathcal{D}$) $=$ maximum bag size $-1$.

## Graphs with width-bounded decompositions

- Paths and trees have width-1 decompositions,
- Series-parallel graphs have width-2 decompositions.

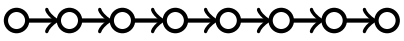
# Different tree decompositions are possible.

## Flexibility of tree decompositions

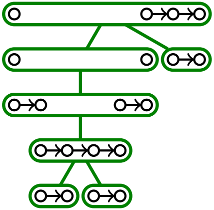
- Many width-optimal decompositions exist in general.
- More apparent with a slightly higher width.

## Width-3 decompositions of a string structure

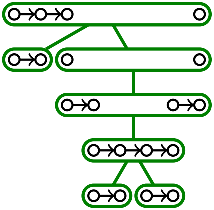
string structure $\mathcal{S}$



decomposition $\mathcal{D}_1$



decomposition $\mathcal{D}_2$



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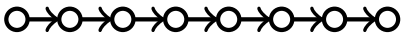
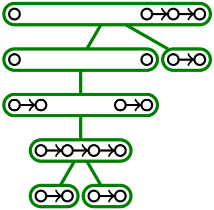
# Our logic adds quantifying decompositions.

## Structures expanded by decompositions

Combining string structure $\mathcal{S}$ and decomposition $\mathcal{D}$ into a structure like $(\mathcal{S}, \mathcal{D}) =$ 



## Decomposition MSO (DMSO) for structures

**Syntax** $\psi = \exists^{\text{width} \le w} D \, [\varphi]$ with MSO $\varphi$ and $w \in \mathbb{N}$.

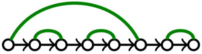
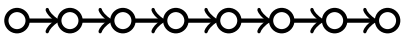
**Semantics** $\mathcal{A} \models \psi$ if $(\mathcal{A}, \mathcal{D}) \models \varphi$ for a width-$w$ tree decomposition $\mathcal{D}$ of $\mathcal{A}$.

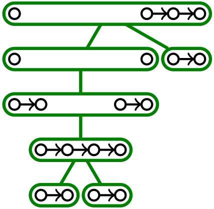
# DMSO on strings captures context-freeness.

## Theorem

*The* DMSO*-definable properties of strings are the* *context-free* *languages.*

## Proof relates matchings and decompositions.

- Define CFLs in terms of “∃ Match MSO”-logic.
- From  $\models \varphi$ to  $\models \varphi'$.



- From $(\mathcal{S}, \mathcal{D}) \models \varphi'$ to $(\mathcal{S}, \mathcal{M}) \models \varphi$. □

# Higher width means higher expressivity.

## Recall: Syntax of DMSO-logic

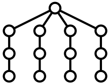
DMSO-formulas are $\psi = \exists^{\text{width} \leq w} D \left[\varphi\right]$ with width $w \in \mathbb{N}$

## Theorem

*For every $w \geq 3$ and $w' \leq w - 2$, formulas with width $w$ are more expressive than formulas with width $w'$.*

## Proving subset and not-equal relation.

"$\subseteq$"
- Width-$w$ cover width-$w'$ decompositions.
- Restricting to width-$w'$ is MSO-definable.

"$\neq$"
- Width $w = 4$ allows to define subdivided stars with 4 paths of equal length like  , but
- is not possible with $w = 2$ (via Ehrenfeucht-Fraïssé games). $\square$

# Satisfiability for DMSO-formulas is decidable.

## Theorem

*Testing whether a given* DMSO*-formula* $\psi = \exists^{\text{width} \leq w} D\left[\varphi\right]$ *has an* $\mathcal{A}$ *with* $\mathcal{A} \models \psi$ *is decidable.*

## Proof is based on Seese's Theorem.

- To test if there is $\mathcal{A}$ with $\mathcal{A} \models \psi$, we test if there is $\mathcal{A}$ and width-$w$ $\mathcal{D}$ with $(\mathcal{A}, \mathcal{D}) \models \varphi$.
- Expansion $(\mathcal{A}, \mathcal{D})$ has tree width at most $w + 2$.
- Thus, satisfiability for subformula $\varphi$ is decidable [Seese, 1991].
- In turn, satisfiability for formula $\psi$ is decidable. $\square$

# Model Checking DMSO-formulas in PTIME.

## Theorem

*Each* DMSO*-definable property is decidable in polynomial time.*

## Proof is based on dynamic programming.

- Let $\psi = \exists^{\text{width} \leq w} D \left[\varphi\right]$ be a formula for the property.
- We only need to look at width-*w* decompositions.

**recursion** Try all root bags and all ways of extending it. Strictness implies exponential time.

**dynamic** Bottom-up evaluation of recursive calls. Normal condition implies polynomial time. □

# Conclusion

## Summary

- DMSO-logic is proposed to capture the notion of context-freeness for graphs.
- Basic expressibility and closure results.
- Satisfiability is decidable, model checking in PTIME.

## Outlook

- Compare DMSO-logic to grammar-based notions of context-freeness for graphs?
- Model checking …
  … fixed-parameter tractable with parameter $|\psi|$?
  … in LOGCFL $\subseteq$ NC$^2$ $\subseteq$ PTIME for each fixed $\psi$?

# References I

Büchi, J. R. (1960).
Weak second-order arithmetic and finite automata.
*Math. Logic Quart.*, 6(1–6):66–92.

Courcelle, B. (1990).
Graph rewriting: An algebraic and logic approach.
In *Handbook of Theoretical Computer Science, Volume B*, pages 193–242. Elsevier and MIT Press.

Elgot, C. C. (1961).
Decision problems of finite automata design and related arithmetics.
*Transactions of the American Mathematical Society*, 98(1):21–51.

# References II


Seese, D. (1991).
The structure of the models of decidable monadic theories of graphs.
*Annals of pure and applied logic*, 53(2):169–195.

Stockmeyer, L. J. (1976).
The polynomial-time hierarchy.
*Theoretical Computer Science*, 3(1):1 – 22.

Trakhtenbrot, B. A. (1961).
Finite automata and logic of monadic predicates.
*Doklady Akademii Nauk SSSR*, 140:326–329.
In Russian.