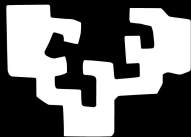


Quantified Constraint Satisfaction on Monoids

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The constraint satisfaction problem (CSP)

CSP: decide $\mathbf{B} \stackrel{?}{\models} \exists v_1 \dots \exists v_n \wedge (\text{atoms})$

where...

- ▶ \mathbf{B} finite relational structure
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Examples: 3-colorability ($\mathbf{B} = \mathbf{K}_3$), 2-SAT, Horn-SAT,
algebraic equations problems

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Classification program 2: For each finite struct \mathbf{B} , describe complexity of $\text{QCSP}(\mathbf{B})$

Maybe: Each problem $\text{QCSP}(\mathbf{B})$ in P, NP-complete, or PSPACE-complete...?

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Suppose \mathbf{B} has 2-elt universe.

- ▶ If \mathbf{B} satisfies [one of 6 conditions], $\text{CSP}(\mathbf{B})$ in \mathbf{P} .
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Schaefer's dichotomy on QCSP (full proof in CKS '01):

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- ▶ Over all 3-element structures \mathbf{B} ...
 - ▶ Classification of $\text{CSP}(\mathbf{B})$: known (Bulatov '06)
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pass from **set of relations** \rightsquigarrow **set of operations (algebra)**

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So, in CSP, can focus on classifying idempotent algebras

In QCSP: open issue!

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Def: A **semigroup** (S, \circ) consists of a set S and an associative binary operation \circ

Assumption: S is always finite

Def: A **monoid** is a semigroup with an identity element (ie, an element e such that $e \circ x = x \circ e = x$)

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In the rest of the talk, we study the CSP and QCSP parameterized by an algebra \mathbb{A}

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In the rest of the talk, we study the CSP and QCSP parameterized by an algebra \mathbb{A}

For each algebra \mathbb{A} , can define $\text{CSP}(\mathbb{A})$, $\text{QCSP}(\mathbb{A})\dots$

Main theorem

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Let S be a semigroup.

- ▶ The problem $\text{CSP}(S)$ is the CSP restricted to structures \mathbf{B} that are preserved by $S = (S, \circ)$.
- ▶ The problem $\text{QCSP}(S)$ defined analogously.

Def: \mathbf{B} preserved by $S = (S, \circ)$ if, for each relation $R^{\mathbf{B}}$:
 $(a_1, \dots, a_k), (b_1, \dots, b_k) \in R^{\mathbf{B}}$ implies $(a_1 \circ b_1, \dots, a_k \circ b_k) \in R^{\mathbf{B}}$

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Main Thm: Let S be a monoid.

- ▶ If S satisfies [some condition], then $\text{QCSP}(S)$ is poly-time decidable.
- ▶ Else, $\text{QCSP}(S)$ is NP-complete.

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Main Thm: Let S be a monoid.

- ▶ If S is a **block group** *and* generated by its **regular** elements, then $\text{QCSP}(S)$ is **poly-time decidable**.
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Def: An elt b is **regular** if $\exists x$ such that $b \circ x \circ b = b$

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Explicitly give a relational struct \mathbf{B} preserved by S where $QCSP(S)$ is **NP-hard**.

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Proof makes use of...

Lemma: Let \mathbb{A} be an algebra;

let \mathbb{B} be a homomorphic image of \mathbb{A} .

Then $QCSP(\mathbb{B})$ **poly-time reduces** to $QCSP(\mathbb{A})$.

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$$\forall \exists \exists^* \bigwedge (\text{atoms})$$

3) For such a sentence $\forall y \exists x_1 \dots \exists x_n \bigwedge (\text{atoms})$, show that if y is set equal to a regular element, can be polytime decided if \exists variables can be set to satisfy \bigwedge

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Note: There exists a semilattice S such that QCSP(S) is **PSPACE-complete** (Börner et al., '09)

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