Quantified Constraint Satisfaction on Monoids

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The constraint satisfaction problem (CSP)

CSP: decide $\mathbf{B} \overset{?}{\models} \exists v_1 \ldots \exists v_n \bigwedge (\text{atoms})$
where...

- $\mathbf{B}$ finite relational structure
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CSP(**B**): given existential conjunctive ϕ,
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Examples: 3-colorability ($\mathbf{B} = \mathbf{K}_3$), 2-SAT, Horn-SAT,
algebraic equations problems

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Maybe: Each problem QCSP(**B**) in P, NP-complete, or PSPACE-complete...?

Schaefer's dichotomies

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Suppose **B** has 2-elt universe.

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Schaefer's dichotomy on QCSP (full proof in CKS '01):
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  - Classification of QCSP(**B**): open
- Over all 3-element structures **B**...
  - Classification of CSP(**B**): known (Bulatov ’06)
  - Classification of QCSP(**B**): open

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pass from set of relations $\rightsquigarrow$ set of operations (algebra)

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So, in CSP, can focus on classifying idempotent algebras

In QCSP: open issue!

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Assumption: S is always finite

Def: A monoid is a semigroup with an identity element (ie, an element e such that $e \circ x = x \circ e = x$)

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For each algebra $\mathbb{A}$, can define $\text{CSP}(\mathbb{A})$, $\text{QCSP}(\mathbb{A})$...

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Let $\mathbb{S}$ be a semigroup.

- The problem CSP($\mathbb{S}$) is the CSP restricted to structures **B** that are preserved by $\mathbb{S} = (S, \circ)$.
- The problem QCSP($\mathbb{S}$) defined analogously.

Def: **B** preserved by $\mathbb{S} = (S, \circ)$ if, for each relation $R^{\mathbf{B}}$:
$(a_1, \ldots, a_k), (b_1, \ldots, b_k) \in R^{\mathbf{B}}$ implies $(a_1 \circ b_1, \ldots, a_k \circ b_k) \in R^{\mathbf{B}}$

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Known Thm (Chen '08): When $\mathbb{S}$ is a monoid, QCSP$(\mathbb{S})$ is in NP.

Main Thm: Let $\mathbb{S}$ be a monoid.

- If $\mathbb{S}$ satisfies [some condition], then QCSP$(\mathbb{S})$ is poly-time decidable.
- Else, QCSP$(\mathbb{S})$ is NP-complete.

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- If $\mathbb{S}$ is a block group *and* generated by its regular elements, then QCSP$(\mathbb{S})$ is poly-time decidable.
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Def: An elt b is regular if $\exists x$ such that $b \circ x \circ b = b$

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Explicitly give a relational struct **B** preserved by $\mathbb{S}$ where QCSP$(\mathbb{S})$ is NP-hard.
The struct **B** has a single 4-ary relation.

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Proof makes use of...

Lemma: Let $\mathbb{A}$ be an algebra;
let $\mathbb{B}$ be a homomorphic image of $\mathbb{A}$.
Then QCSP$(\mathbb{B})$ poly-time reduces to QCSP$(\mathbb{A})$.

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3) For such a sentence $\forall y \exists x_1 \ldots \exists x_n \bigwedge \text{(atoms)}$,
show that if y is set equal to a regular element,
can be polytime decided if $\exists$ variables can be set to satisfy $\bigwedge$

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Note: There exists a semilattice $\mathbb{S}$ such that
QCSP$(\mathbb{S})$ is PSPACE-complete (Börner et al., '09)

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end [fin]