

FAKULTÄT FÜR INFORMATIK

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Monadic second order finite satisfiability and unbounded tree-width

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Deciding finite satisability

- **Theorem** [Trakthenbrot 1950]
- The finite satisfiability problem of First Order logic FO is undecidable.

- Approaches for decidability
- 1) Restricting the logic
- 2) Restricting the structures

Restricting the logic

Decidable fragments of FO:

- Prefix classes of FO
- FO²: two-variable fragment of FO
- C²: FO² with counting quantifiers
- GF: guarded fragment of FO
- Modal logics
- Description logics
- Temporal logics

Restricting the structures

Theorem [Courcelle 1990]

The finite satisfiability problem of Monadic Second Order Logic MSO by graphs of bounded tree-width is decidable.

Theorem [Seese 1991]

The finite satisfiability of MSO sentences over any class of graphs of unbounded tree-width is undecidable.

Restricting the logic and adding structures

FO² / C²

symbol s interpreted as

- linear order
- pre-order
- equivalence relation
- weak transitive closure
- tree

FO² + special relation

[Otto 2001] [Bojanczyk, Muscholl, Schwentik, Segoufin, David 2006] [Schwentick, Zeume 2010] [Kieronski 2011]

C² + **two trees** [Charatonik, Witkowski 2013]

C² + two trees

Definition ($T(s_1, s_2)$)

The class of finite structures in which s_1 , s_2 are interpreted as directed trees.

Theorem [Charatonik, Witkowski 2013] Satisfiability of C_2 over $T(s_1, s_2)$ is NEXPTIME-complete.

Our Result

Given $\alpha_{MSO} \in MSO(\sigma_{MSO})$ $\alpha_{C2} \in C^2(\sigma_{C2})$ $k \in N$

<u>Problem</u>: Is $\alpha_{MSO} \wedge \alpha_{C2}$ satisfiable by a finite structure M with treewidth(M|_{\sigma_{MSO}}) \le k?

Theorem [Kotek, Veith, Z.] The above problem is decidable.

Generalization of the results

- Courcelle 1990
- Charatonik,
 Witkowski 2013
 (for one tree)

Our Result

Given
$$\alpha_{MSO} \in MSO(\sigma_{MSO})$$

 $\alpha_{C2} \in C^2(\sigma_{C2})$
 $k \in N$

<u>Problem:</u> Is $\alpha_{MSO} \wedge \alpha_{C2}$ satisfiable by a finite structure M with treewidth(M| $_{\sigma_{MSO}}$) \leq k?

Theorem [Kotek, Veith, Z.] The above problem is decidable.



Motivation: Verification of Data Structures

MSO

Two disjoint data structures

- A binary tree whose leaves are chained
- A cyclic list

(tree-width \leq 3)

C²

- Every tree leaf has exactly one outgoing edge to the cyclic list
- The cyclic list nodes have incoming edges only from the tree leaves

(unbounded tree-width)



Application: MSO with cardinalities

• MSO^{card} extends MSO by cardinality constraints $|X_1| + ... + |X_r| \le |Y_1| + ... + |Y_t|$

Theorem [Klaedtke and Rue 2003] MSO^{card} satisfiability on finite structures of bounded tree-width is undecidable.

Application: MSO with cardinalities

- MSO^{\exists card} is the fragment of MSO^{card}, where $\exists \overline{X}. \varphi$, and only \overline{X} occur in cardinality constraints
- Example: $\exists X_1$. $|X_1| = |\neg X_1| \land \varphi$



Theorem [Kotek, Veith,Z] MSO^{∃card} satisfiability on finite structures of bounded tree-width is decidable.

Application: MSO with cardinalities

- MSO^{\exists card} is the fragment of MSO^{card}, where $\exists \overline{X}. \varphi$, and only \overline{X} occur in cardinality constraints</sup>
- Example: $\exists X_1$. "F is a bijection from X_1 to $\neg X_1$ " $\land \varphi$



Theorem [Kotek, Veith,Z] MSO^{∃card} satisfiability on finite structures of bounded tree-width is decidable.

Outline of the Proof

- 1. Our Separation Theorem: Reduction to the case that σ_{MSO} and σ_{C2} only share unary relation symbols Types, Scott-Normal form, Colorings
- 2. From structures of bounded tree-width to labeled binary trees
 - Translation schemes
- 3. From MSO to C^2
 - Feferman-Vaught theorem on labeled trees, Hintikka sentences
- 4. Decidability of C^2 + binary tree
 - Charatonik and Witkowski, LICS 2013

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Separation Theorem

Given $k \in N$ and

 $\alpha_{MSO} \in MSO(\sigma_{MSO}), \ \alpha_{C2} \in C^2(\sigma_{C2}),$ there effectively are

 $\alpha'_{MSO} \in MSO(\sigma'_{MSO}), \alpha'_{C2} \in C^2(\sigma'_{C2}),$ with $\sigma'_{MSO} \cap \sigma'_{C2} = , unary relation symbols",$ such that

 $\alpha_{MSO} \wedge \alpha_{C2}$ is satisfiable by M with treewidth(M| $_{\sigma_{MSO}}$) $\leq k$ iff

 α' MSO $\wedge \alpha'_{C2}$ is satisfiable by M with treewidth(M| $_{\sigma'}$ MSO) $\leq k$. (All structures are finite.)

Shared Binary Symbols



Because of treewidth($M|_{\sigma}MSO$) \leq k we can assume σ_{MSO} = some unary relations symbols + the binary relations symbols $R_1, ..., R_k$ and

 $R_1, ..., R_k$ are interpreted by functional relations.



Justification:

- 1. We can axiomatize that $R_1, ..., R_k$ are functional.
- 2. Other shared symbols can be simulated by $R_1, ..., R_k$ and unary relation symbols.

Main Idea of Separation Theorem

 $\mathbf{M} \xrightarrow{\mathbf{R}_{1}}_{\mathbf{R}_{2}} \xrightarrow{\mathbf{R}_{1}}_{\mathbf{R}_{3}} \xrightarrow{\mathbf{R}_{1}}_{\mathbf{R}_{2}}$

We introduce fresh copies $\overline{R_1},...,\overline{R_k}$ of $\mathsf{R}_1,...,\mathsf{R}_k.$

We obtain $\overline{\alpha_{MSO}}$ from α_{MSO} by replacing every R_i with its copy $\overline{R_i}$.

 $\overline{\alpha_{MSO}}$ and α_{C2} don't share binary relation symbols!

Problem:

The interpretations of $R_1, ..., R_k$ and $\overline{R_1}, ..., \overline{R_k}$ do not need to agree in a model M of $\overline{\alpha_{MSO}} \wedge \alpha_{C2}$.

Main Idea: Swapping Edges

M is a model of $\overline{\alpha_{MSO}} \wedge \alpha_{C2}$



<u>Idea:</u>

Swap edges until the interpretations of $R_1, ..., R_k$ and $\overline{R_1}, ..., \overline{R_k}$ do agree!

We will axiomatize conditions which guarantee that such a sequence of edge swaps always exists.

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Scott-Normal Form

Every C² formula is equi-satisfiable to a forumla $\forall x, y. \phi \land \Lambda_i \forall x \exists^{=1} y. S_i(x, y),$ where ϕ is quantifier-free.

In this talk, we assume $\begin{array}{l} \text{binary}(\sigma_{\text{C2}}) = \{\text{R}_1, \dots, \text{R}_k, \text{S}\} \\ \text{and } \alpha_{\text{C2}} \in \text{C}_2(\sigma_{\text{C2}}) \text{ is} \\ \forall x, y. \ \phi \land \ \forall x \ \exists^{=1}y. \ \text{S}(x, y), \\ \text{which already shows all difficulties.} \end{array}$

Edge Swaps: Invariance of Validity



$$\forall x, y. \phi \land \forall x \exists^{=1}y. S(x, y)$$

emma: Edge Swap does not affect validity.



Lemma: Edge Swap does not affect validity.

Axiomatizing Edge-Types I

fresh unary relation symbols



<u>We axiomatize:</u> If a node has an outgoing edge labeled by R_1 , R_2 , or an outgoing edge labeled by $\overline{R_1}$, $\overline{R_2}$. then the node is labeled by $P_{\{R_1,R_2\}}$

We axiomatize:

A node labeled by the unary relation P_{{R1,R2} has an outgoing edge labeled by R₁ and R₂ and an outgoing edge labeled by $\overline{R_1}$ and $\overline{R_2}$

Axiomatizing Edge-Types II



Problem:

Edge swap will violate the functionality of S.

Axiomatizing Edge-Types II



Insight: The axiomatized edge-types also need to contain

- the relation S,
- the inverse relations R_1^{-1} ,..., R_k^{-1} , S^{-1} .

Axiomatizing Edge-Types II



Colorings



<u>Problem:</u> Edge swap will violate the functionality of R₃

Colorings



Problem:

Edge swap will violate the functionality of R_3

Idea:

We add the unary relations satsified by start- and end-node of the edges (= 1-types) to the P-predicates

Colorings



Problem:

Edge swap will violate the functionality of R₃

Idea:

We add the unary relations satsified by start- and end-node of the edges (= 1-types) to the P-predicates

<u>Lemma</u>: We can axtiomatize that, if $\bigcirc_A \bigcirc_B \bigcirc_B \bigcirc_B$, with A,B $\in \{R_1,...,R_k,S\}$, then \bigcirc and \bigcirc satisfy different unary relations.

<u>Lemma:</u> Every graph of bounded out-degree can be colored.

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From structures of bounded treewidth to labeled binary trees

Given $k \in N$ and

 $\alpha_{MSO} \in MSO(\sigma_{MSO}), \alpha_{C2} \in C^2(\sigma_{C2}),$ with $\sigma_{MSO} \cap \sigma_{C2} = "unary relation symbols", <math>s \notin \sigma_{MSO}$, there effectiely are

 $\begin{array}{l} \alpha ^{\prime} \mathrm{MSO} \in \mathrm{MSO}(\sigma ^{\prime} \mathrm{MSO}), \, \alpha ^{\prime} \mathrm{C2} \in \mathrm{C}^{2}(\sigma ^{\prime} \mathrm{C2}), \\ \text{with } \sigma ^{\prime} \mathrm{MSO} \cap \sigma ^{\prime} \mathrm{C2} = \, \text{,unary relation symbols}^{\prime\prime}, \, \mathbf{s} \in \boldsymbol{\sigma} ^{\prime} \mathrm{MSO}, \\ \text{such that} \end{array}$

 $\alpha_{MSO} \wedge \alpha_{C2}$ is satisfiable by M with treewidth(M| $_{\sigma_{MSO}}$) $\leq k$

iff

 $\alpha'_{MSO} \wedge \alpha'_{C2}$ is satisfiable by M and s is interpreted by a binary tree. (All structures are finite.) From structures of bounded treewidth to labeled binary trees Because we are interested in finite structures M with

treewidth($M|_{\sigma}MSO$) $\leq k$, we can assume that the relations $R_1,...,R_k$ are interpreted as a k-tree.

Idea:

We encode a k-tree by a binary tree using a translation scheme.

Encoding a k-tree by a binary tree



",1,...,k and N partition the universe"

In α and β we replace $\exists x. \phi \text{ by } \exists x. \neg N(x) \land \phi$ $\forall x. \phi \text{ by } \exists x. \neg N(x) \rightarrow \phi$

In α we replace $R_i(x,y)$ by the formula $\theta_i(x,y) =$

"x labeled by j" and

((x=y and "there is no ancestor labeled by i+i mod k+1") or

("y is an ancestor of x labeled by j+i mod k+1" and

"there is no node between x and y labeled by j+i mod k+1"))

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From MSO to C²

Given

$\alpha_{MSO} \in MSO(\sigma_{MSO}),$ there effectiely is $\alpha'_{MSO} \in \mathbb{C}^{2}(\sigma'_{MSO}),$ such that

 α_{MSO} is satisfiable by a finite structure M, where s is interpreted as a binary tree,

iff

 α'_{MSO} is satisfiable by a finite structure M, where s is interpreted as a binary tree.

Hintakka Sentences

Let $q \in N$. There is a finite set HIN of MSO(σ) sentences of quantifier-rank q such that:

- The models of the formulae in HIN are mutually disjoint.
- Every $\alpha \in MSO(\sigma)$ of quantifier-rank q is equivalent to a (finite) disjunction $V_i \varphi_i$, with $\varphi_i \in HIN$.

From MSO to C²



Feferman-Vaught Theorem:

There effectively is a function $\varphi_k, \varphi_j \rightarrow \varphi_h$ such that a tree satisfies the Hintakka sentence φ_h iff the subtrees at children of the root satisfy the Hintakka sentences φ_k, φ_j .

Reduction:

We can encode the function $\varphi_k, \varphi_j \rightarrow \varphi_h$ by a C² formula. We require the tree root to satisfy the formula $V_i C_{\varphi_i}$, where $V_i \varphi_i$ is equivalent to $\alpha_{MSO} \in MSO(\sigma_{MSO})$.

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Thanks for your attention!