Monadic second order finite satisfiability and unbounded tree-width

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Joint work with Tomer Kotek, Helmut Veith

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Deciding finite satisability

Theorem [Trakthenbrot 1950]

The finite satisfiability problem of First Order logic FO is undecidable.

Approaches for decidability

1) Restricting the logic
2) Restricting the structures

Restricting the logic

Decidable fragments of FO:

- Prefix classes of FO
- FO^2: two-variable fragment of FO
- C^2: FO^2 with counting quantifiers
- GF: guarded fragment of FO
- Modal logics
- Description logics
- Temporal logics

Restricting the structures

Theorem [Courcelle 1990]

The finite satisfiability problem of Monadic Second Order Logic MSO by graphs of bounded tree-width is decidable.

Theorem [Seese 1991]

The finite satisfiability of MSO sentences over any class of graphs of unbounded tree-width is undecidable.

Restricting the logic and adding structures

FO^2 / C^2

+

symbol **s** interpreted as
- linear order
- pre-order
- equivalence relation
- weak transitive closure
- tree

FO^2 + **special relation**
[Otto 2001]
[Bojanczyk, Muscholl, Schwentik, Segoufin, David 2006]
[Schwentick, Zeume 2010]
[Kieronski 2011]

C^2 + **two trees**
[Charatonik, Witkowski 2013]

C^2 + two trees

Definition ($T(s_1, s_2)$)
The class of finite structures in which s_1, s_2 are interpreted as directed trees.

Theorem [Charatonik, Witkowski 2013**]**
Satisfiability of C_2 over $T(s_1, s_2)$
is NEXPTIME-complete.

Our Result

Given $\alpha_{MSO} \in MSO(\sigma_{MSO})$

$\alpha_{C2} \in C^2(\sigma_{C2})$

$k \in N$

<u>Problem:</u> Is $\alpha_{MSO} \wedge \alpha_{C2}$ satisfiable by a finite structure M with treewidth$(M|_{\sigma_{MSO}}) \leq k$?

Theorem [Kotek, Veith, Z.]

The above problem is decidable.

Generalization of the results

- Courcelle 1990
- Charatonik, Witkowski 2013 (for one tree)

Our Result

Given $\alpha_{MSO} \in MSO(\sigma_{MSO})$

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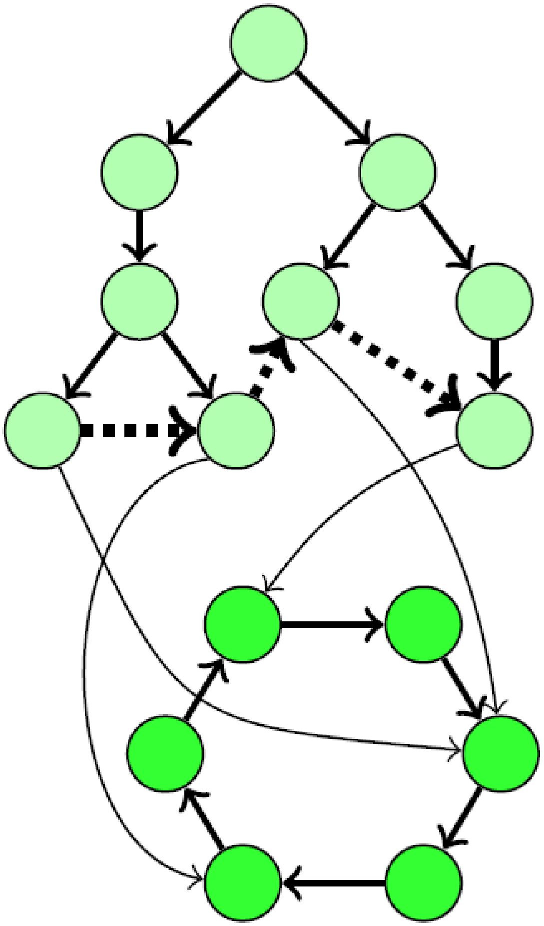
<u>Problem:</u> Is $\alpha_{MSO} \wedge \alpha_{C2}$ satisfiable by a finite structure M with treewidth($M|_{\sigma_{MSO}}$) ≤ k?



Theorem [Kotek, Veith, Z.]

The above problem is decidable.

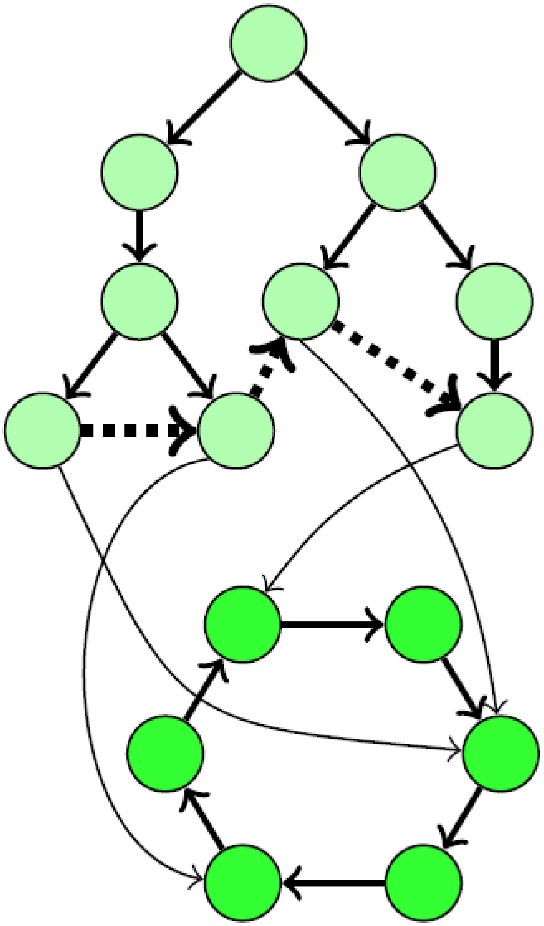
Motivation: Verification of Data Structures

MSO

Two disjoint data structures

- A binary tree whose leaves are chained
- A cyclic list

(tree-width ≤3)



C^2

- Every tree leaf has exactly one outgoing edge to the cyclic list
- The cyclic list nodes have incoming edges only from the tree leaves

(unbounded tree-width)

Application: MSO with cardinalities

- MSO^{card} extends MSO by **cardinality constraints**

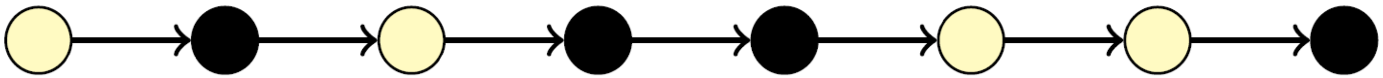
$$|X_1| + \ldots + |X_r| \leq |Y_1| + \ldots + |Y_t|$$

Theorem [Klaedtke and Rue 2003]

MSO^{card} satisfiability on finite structures of bounded tree-width is undecidable.

Application: MSO with cardinalities

- $MSO^{\exists card}$ is the fragment of MSO^{card}, where $\exists \overline{X}. \varphi$, and only $\overline{X}$ occur in cardinality constraints
- Example: $\exists X_1 .\ |X_1| = |\neg X_1| \wedge \varphi$

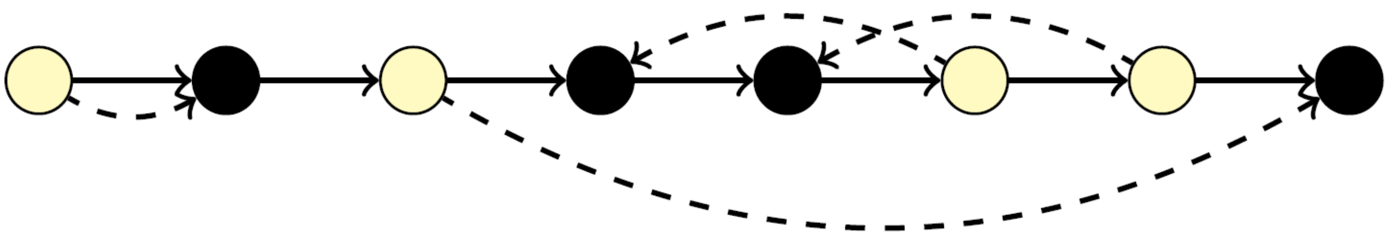


Theorem [Kotek, Veith,Z]
$MSO^{\exists card}$ satisfiability on finite structures of bounded tree-width is decidable.

Application: MSO with cardinalities

- $MSO^{\exists card}$ is the fragment of MSO^{card}, where $\exists \overline{X}. \varphi$, and only $\overline{X}$ occur in cardinality constraints

- Example: $\exists X_1$. "F is a bijection from X_1 to $\neg X_1$" $\wedge\ \varphi$



Theorem [Kotek, Veith,Z]
$MSO^{\exists card}$ satisfiability on finite structures of bounded tree-width is decidable.

Outline of the Proof

1. Our **Separation Theorem**: Reduction to the case that σ_{MSO} and σ_{C2} only share unary relation symbols
   - Types, Scott-Normal form, Colorings
2. From structures of bounded tree-width to labeled binary trees
   - Translation schemes
3. From MSO to C^2
   - Feferman-Vaught theorem on labeled trees, Hintikka sentences
4. Decidability of C^2 + binary tree
   - Charatonik and Witkowski, LICS 2013

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Separation Theorem

Given $k \in N$ and

$\alpha_{MSO} \in MSO(\sigma_{MSO})$, $\alpha_{C2} \in C^2(\sigma_{C2})$,

there effectively are

$\alpha'_{MSO} \in MSO(\sigma'_{MSO})$, $\alpha'_{C2} \in C^2(\sigma'_{C2})$,

with **$\sigma'_{MSO} \cap \sigma'_{C2}$ = „unary relation symbols“**,

such that

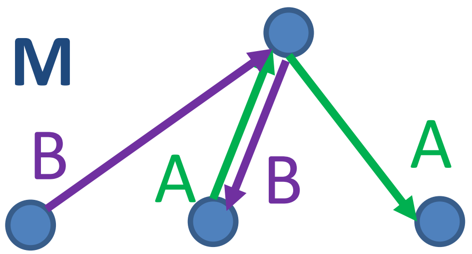
$\alpha_{MSO} \wedge \alpha_{C2}$ is satisfiable by M with treewidth$(M|_{\sigma_{MSO}}) \leq k$

iff

$\alpha'_{MSO} \wedge \alpha'_{C2}$ is satisfiable by M with treewidth$(M|_{\sigma'_{MSO}}) \leq k$.

(All structures are finite.)

Shared Binary Symbols



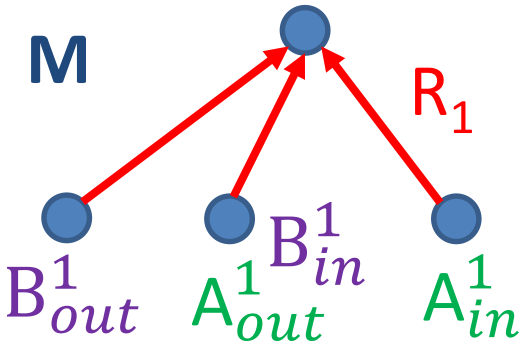
Because of treewidth($M|_{\sigma_{MSO}}$) ≤ k we can assume

σ_{MSO} = some unary relations symbols +

the binary relations symbols $R_1,\ldots,R_k$

and

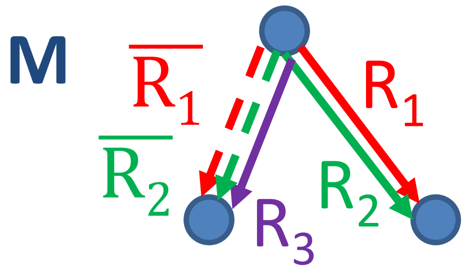
$R_1,\ldots,R_k$ are interpreted by functional relations.



Justification:

1. We can axiomatize that $R_1,\ldots,R_k$ are functional.
2. Other shared symbols can be simulated by $R_1,\ldots,R_k$ and unary relation symbols.

Main Idea of Separation Theorem



We introduce fresh copies $\overline{R_1},\ldots,\overline{R_k}$ of $R_1,\ldots,R_k$.

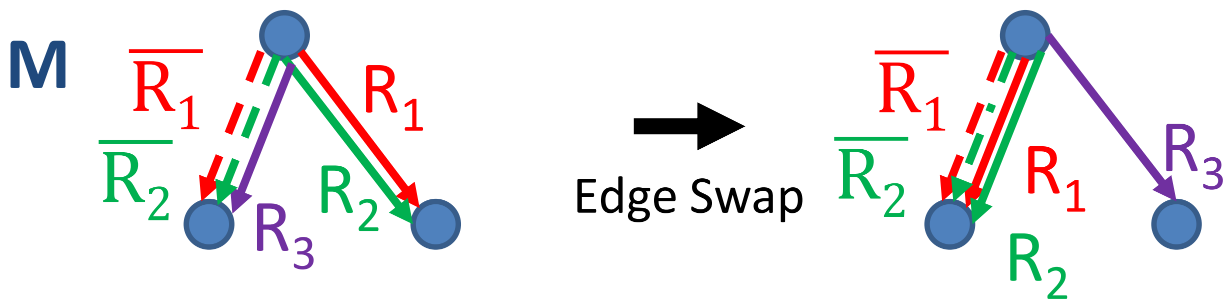
We obtain $\overline{\alpha_{MSO}}$ from α_{MSO} by replacing every R_i with its copy $\overline{R_i}$.

$\overline{\alpha_{MSO}}$ and α_{C2} don‘t share binary relation symbols!

<u>Problem:</u>
The interpretations of $R_1,\ldots,R_k$ and $\overline{R_1},\ldots,\overline{R_k}$ do not need to agree in a model M of $\overline{\alpha_{MSO}} \wedge \alpha_{C2}$.

Main Idea: Swapping Edges

M is a model of $\overline{\alpha_{MSO}} \wedge \alpha_{C2}$



<u>Idea:</u>

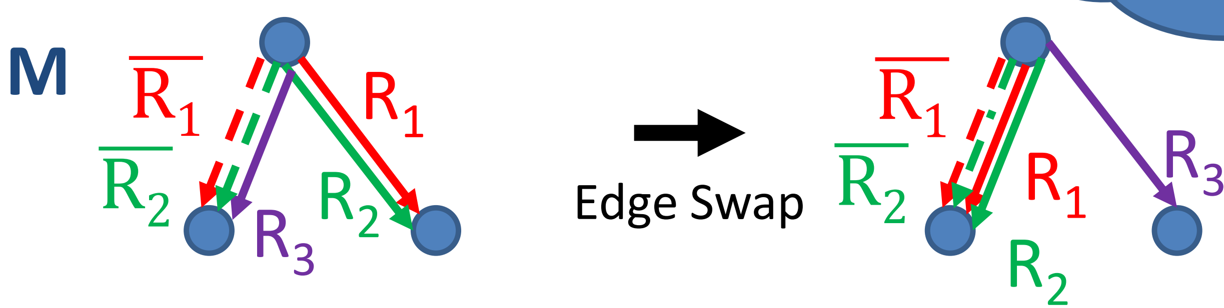
Swap edges until the interpretations of $R_1,\ldots,R_k$ and $\overline{R_1},\ldots,\overline{R_k}$ do agree!

We will axiomatize conditions which guarantee that such a sequence of edge swaps always exists.

Main Idea: Swapping Edges

M is a model of $\overline{\alpha_{MSO}} \wedge \alpha_{C2}$

Validity maintained by edge swap?



<u>Idea:</u>

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Scott-Normal Form

Every C^2 formula is equi-satisfiable to a forumla

$$\forall x,y.\ \varphi \wedge \bigwedge_i \forall x\ \exists^{=1} y.\ S_i(x,y),$$

where φ is quantifier-free.

In this talk, we assume

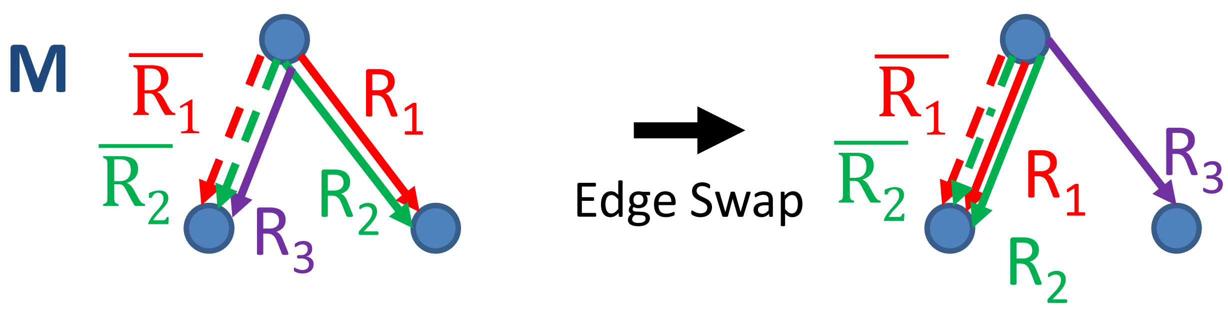
$$\text{binary}(\sigma_{C2}) = \{R_1,\ldots,R_k,S\}$$

and $\alpha_{C2} \in C_2(\sigma_{C2})$ is

$$\forall x,y.\ \varphi \wedge \forall x\ \exists^{=1} y.\ S(x,y),$$

which already shows all difficulties.

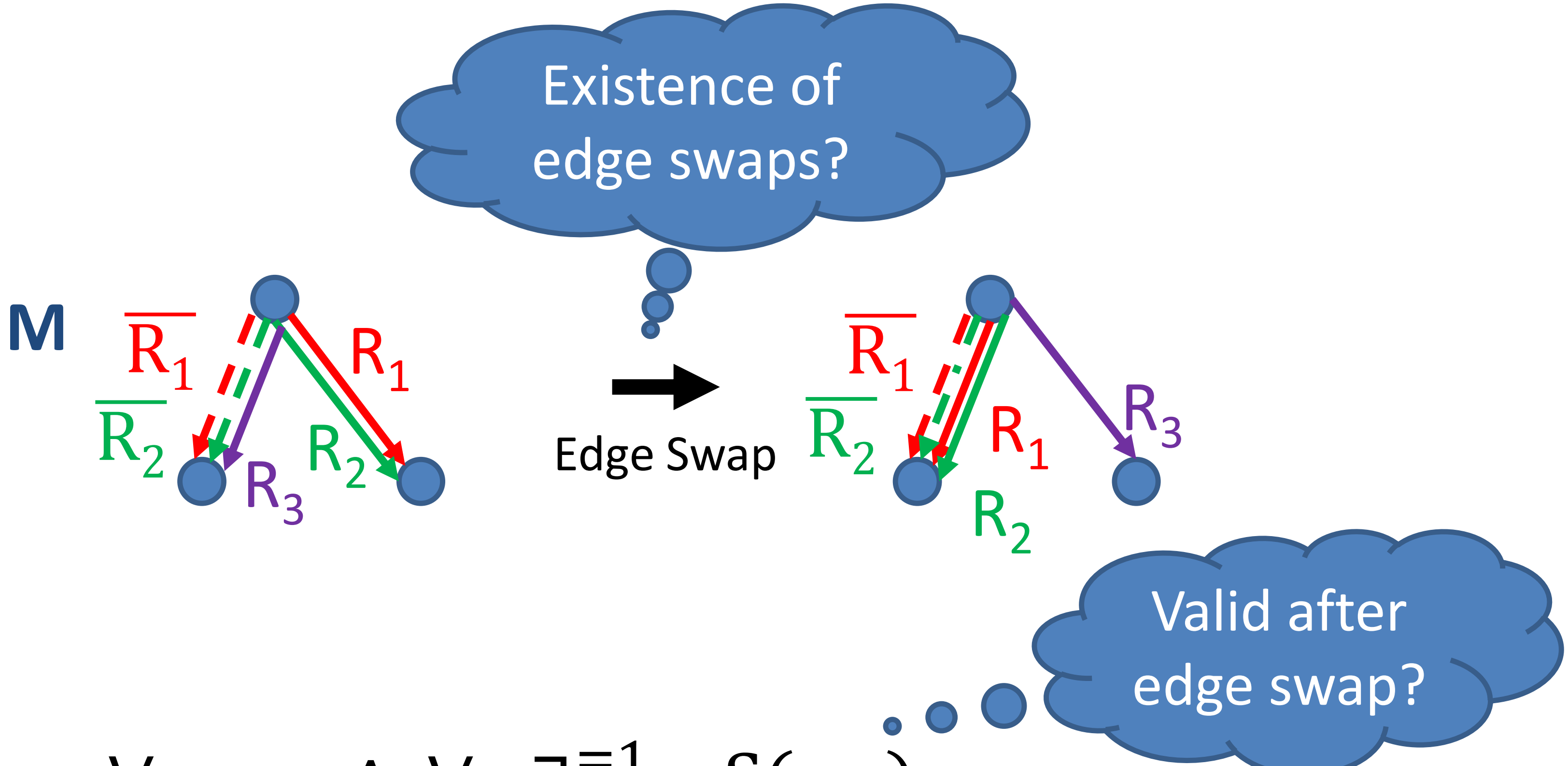
Edge Swaps: Invariance of Validity



$$\underbrace{\forall x,y.\ \varphi}\ \wedge\ \forall x\ \exists^{=1} y.\ S(x,y)$$

<u>Lemma:</u> Edge Swap does not affect validity.

Edge Swaps: Invariance of Validity



$$\underbrace{\forall x,y.\ \varphi}\ \wedge\ \forall x\ \exists^{=1} y.\ S(x,y)$$

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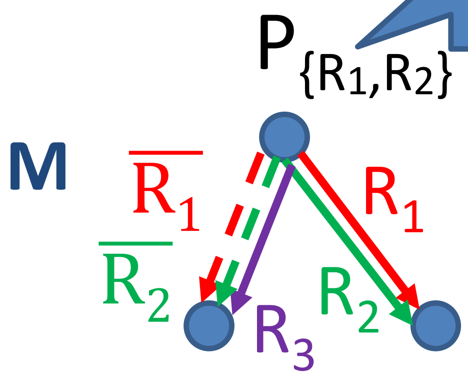
Axiomatizing Edge-Types I

fresh unary relation symbols



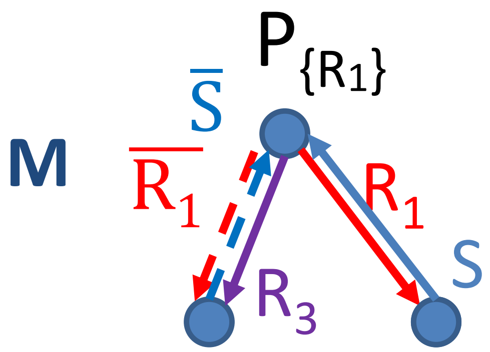
We axiomatize:

If a node has an outgoing edge labeled by R_1, R_2, or an outgoing edge labeled by $\overline{R_1}$, $\overline{R_2}$.
then the node is labeled by $P_{\{R_1,R_2\}}$

We axiomatize:

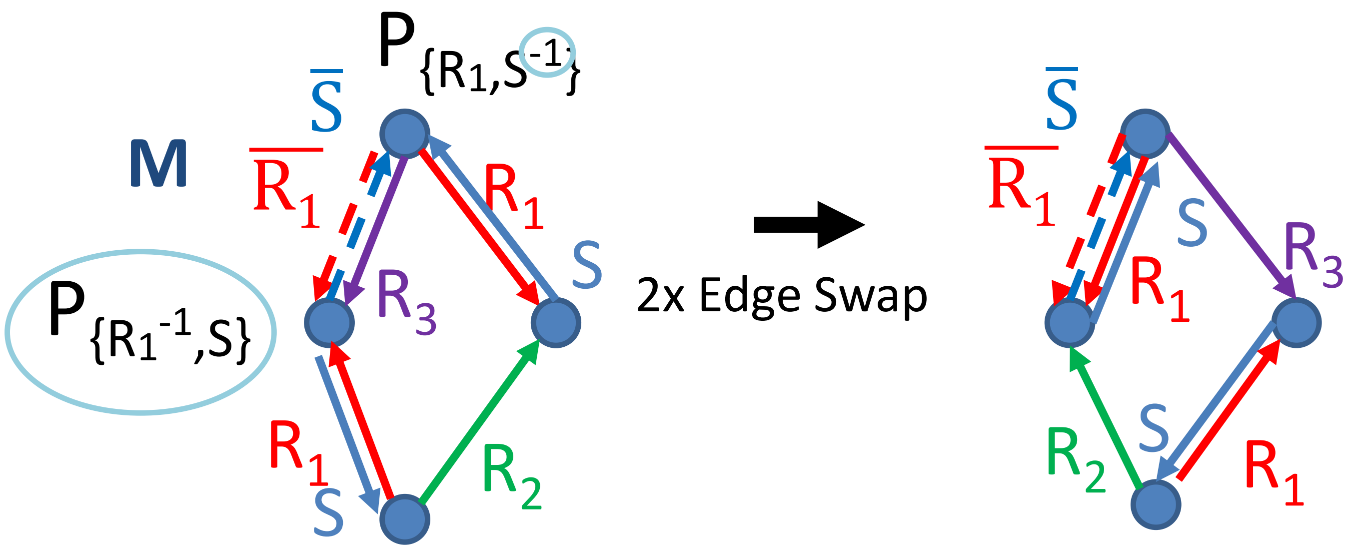
A node labeled by the unary relation $P_{\{R_1,R_2\}}$ has an outgoing edge labeled by R_1 and R_2 and an outgoing edge labeled by $\overline{R_1}$ and $\overline{R_2}$

Axiomatizing Edge-Types II



<u>Problem:</u>
Edge swap will violate the functionality of S.

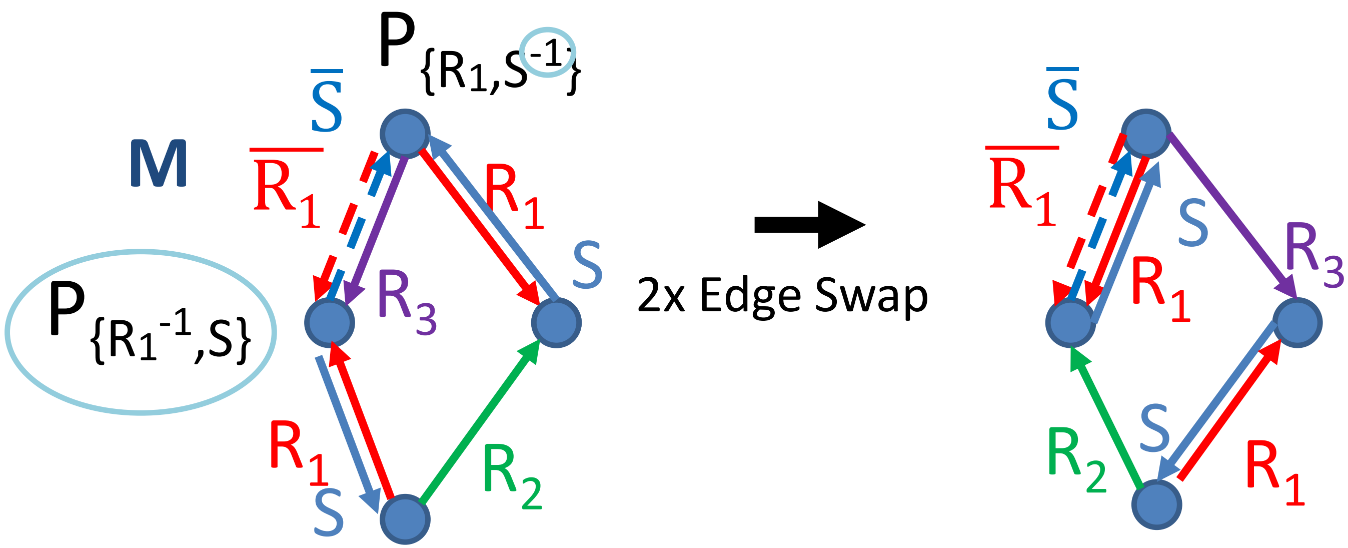
Axiomatizing Edge-Types II



Insight: The axiomatized edge-types also need to contain

- the relation S,
- the inverse relations $R_1^{-1}, \ldots, R_k^{-1}, S^{-1}$.

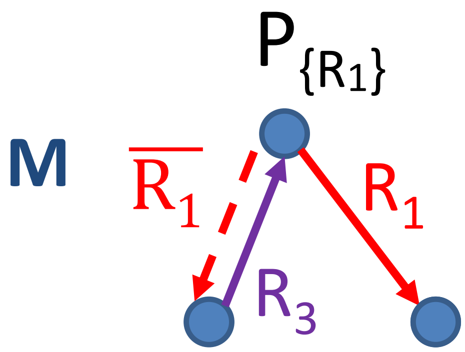
Axiomatizing Edge-Types II



$$\forall x,y.\ \varphi \wedge \underbrace{\forall x\ \exists^{=1} y.\ S(x,y)}$$

<u>Lemma:</u> 2x Edge Swap does not affect functionality.

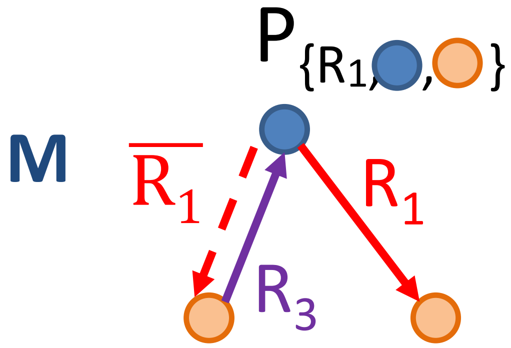
Colorings



Problem:

Edge swap will violate the functionality of R_3

Colorings



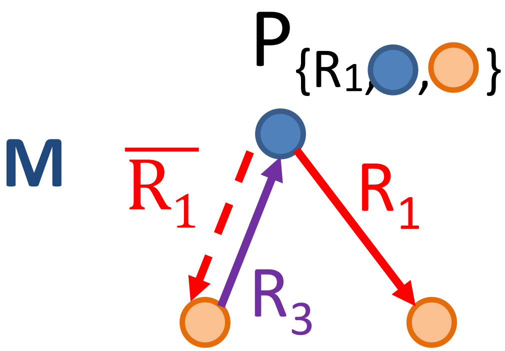
<u>Problem:</u>

Edge swap will violate the functionality of R_3

<u>Idea:</u>

We add the unary relations satsified by start- and end-node of the edges (= 1-types) to the P-predicates

Colorings



Problem:
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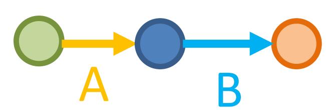
Lemma: We can axtiomatize that, if 

, with A,B ∈ $\{R_1,\ldots,R_k,S\}$, then  and  satisfy different unary relations.

Lemma: Every graph of bounded out-degree can be colored.

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   – Charatonik and Witkowski, LICS 2013

From structures of bounded tree-width to labeled binary trees

Given k ∈ N and

$\alpha_{MSO} \in MSO(\sigma_{MSO})$, $\alpha_{C2} \in C^2(\sigma_{C2})$,

with $\sigma_{MSO} \cap \sigma_{C2}$ = „unary relation symbols“, **s $\notin$ σ_{MSO}**,

there effectiely are

$\alpha'_{MSO} \in MSO(\sigma'_{MSO})$, $\alpha'_{C2} \in C^2(\sigma'_{C2})$,

with $\sigma'_{MSO} \cap \sigma'_{C2}$ = „unary relation symbols“, **s $\in$ σ'_{MSO}**,

such that

$\alpha_{MSO} \wedge \alpha_{C2}$ is satisfiable by M with treewidth($M|_{\sigma_{MSO}}$) ≤ k

iff

$\alpha'_{MSO} \wedge \alpha'_{C2}$ is satisfiable by M and **s is interpreted by a binary tree**.

(All structures are finite.)

From structures of bounded tree-width to labeled binary trees

Because we are interested in finite structures M with

$$\text{treewidth}(M|_{\sigma_{MSO}}) \leq k,$$

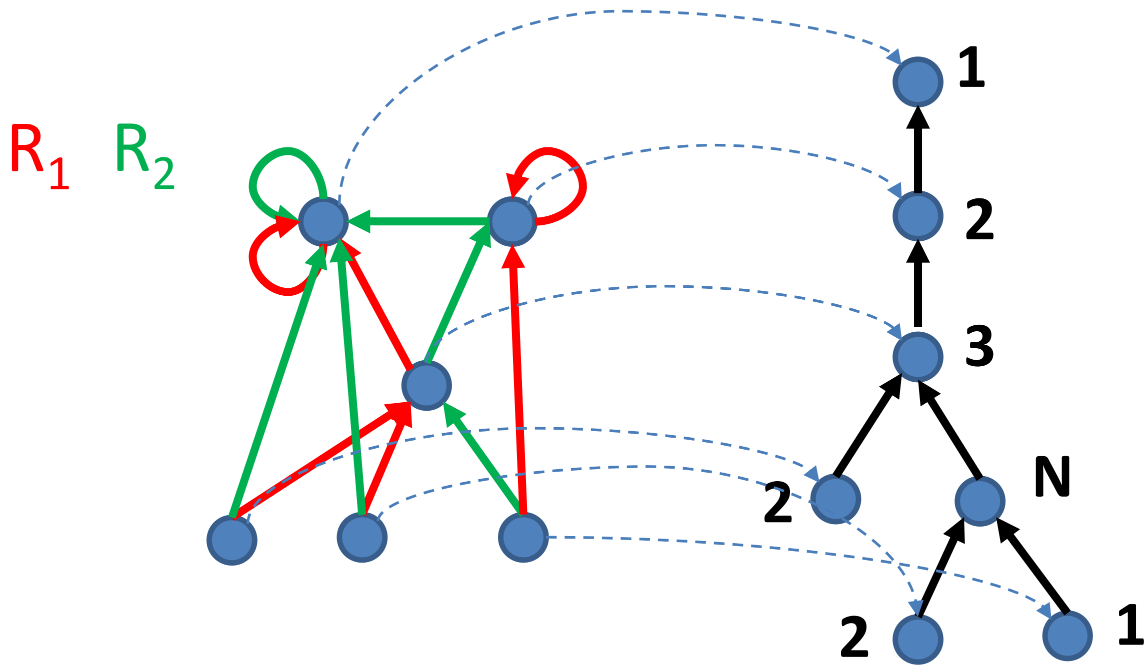
we can assume that the relations

$R_1,\ldots,R_k$ are interpreted as a k-tree.

Idea:

We encode a k-tree by a binary tree using a translation scheme.

Encoding a k-tree by a binary tree



„1,…,k and N partition the universe“

In α and β we replace
$\exists x.\varphi$ by $\exists x.\neg N(x) \land \varphi$
$\forall x.\varphi$ by $\exists x.\neg N(x) \rightarrow \varphi$

In α we replace $R_i(x,y)$ by the formula $\theta_i(x,y)$ =
„x labeled by j“ and
((x=y and „there is no ancestor labeled by j+i mod k+1“) or
(„y is an ancestor of x labeled by j+i mod k+1“ and
„there is no node between x and y labeled by j+i mod k+1“))

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From MSO to C^2

Given

$\alpha_{MSO} \in MSO(\sigma_{MSO})$,

there effectiely is

$\alpha'_{MSO} \in$ **C^2**(σ'_{MSO}),

such that

α_{MSO} is satisfiable by a finite structure M, where s is interpreted as a binary tree,

iff

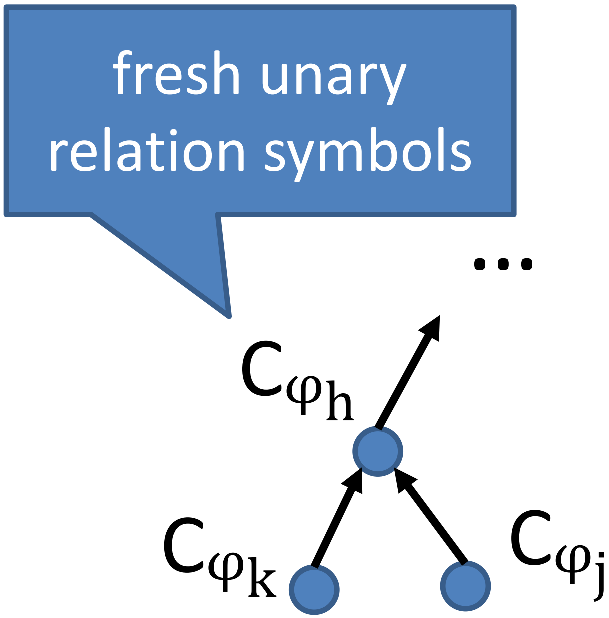
α'_{MSO} is satisfiable by a finite structure M, where s is interpreted as a binary tree.

Hintakka Sentences

Let $q \in N$. There is a finite set HIN of MSO(σ) sentences of quantifier-rank q such that:

- The models of the formulae in HIN are mutually disjoint.
- Every $\alpha \in$ MSO(σ) of quantifier-rank q is equivalent to a (finite) disjunction $\bigvee_i \varphi_i$, with $\varphi_i \in$ HIN.

From MSO to C^2



<u>Feferman-Vaught Theorem:</u>

There effectively is a function $\varphi_k, \varphi_j \rightarrow \varphi_h$ such that a tree satisfies the Hintakka sentence φ_h iff the subtrees at children of the root satisfy the Hintakka sentences φ_k, φ_j.

<u>Reduction:</u>

We can encode the function $\varphi_k, \varphi_j \rightarrow \varphi_h$ **by a C^2 formula**.

We require the tree root to satisfy the formula $\bigvee_i C_{\varphi_i}$, where $\bigvee_i \varphi_i$ is equivalent to $\alpha_{MSO} \in MSO(\sigma_{MSO})$.

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Thanks for your attention!