On the Parallel Complexity of Bisimulation over Finite Systems

CSL 2016

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University of Siegen

joint work with Stefan Göller and Markus Lohrey

Hereditarily finite sets

Are the following two sets equal?

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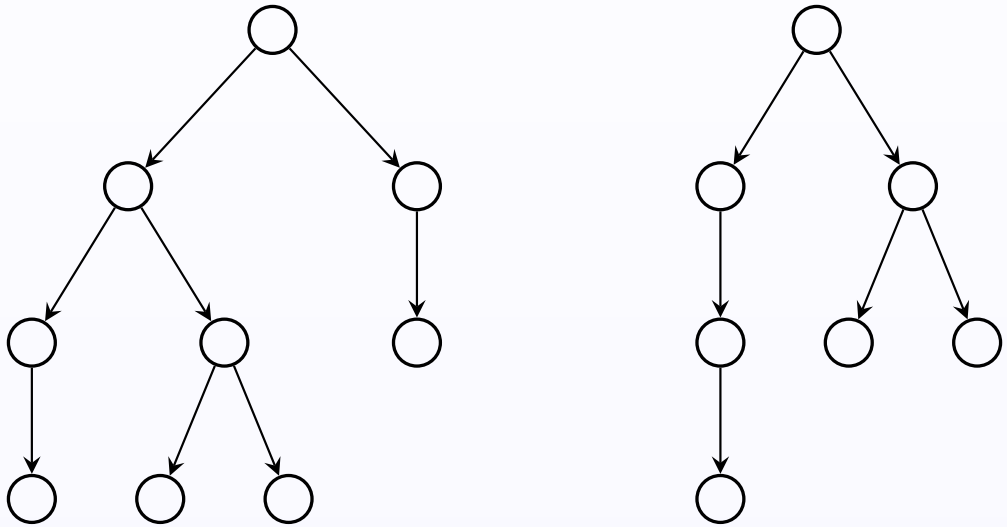

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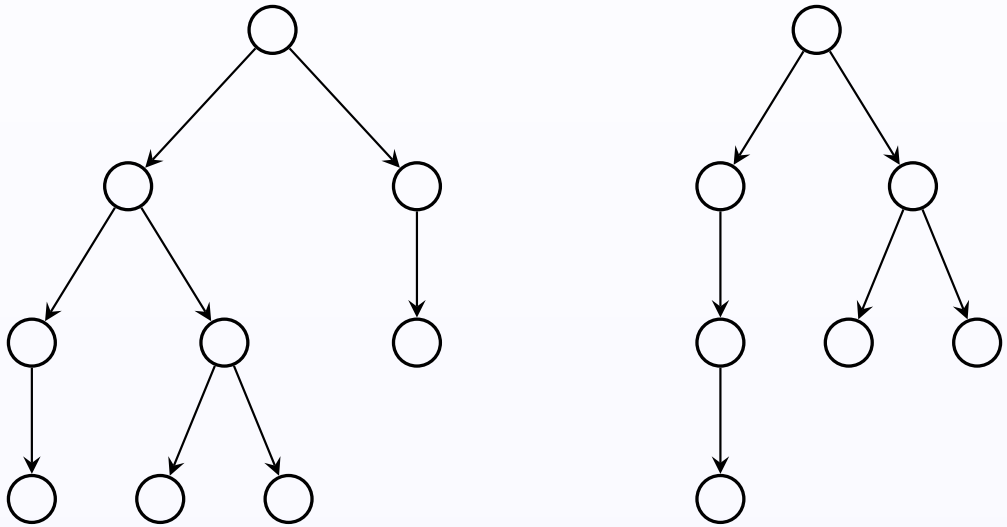

Two sets are equal iff the corresponding trees are bisimilar.

Starting point: Courcelle's theorem

Theorem [Courcelle, 1990]

For every MSO-sentence ψ and every $k \in \mathbb{N}$, one can decide in linear time whether a given structure $\mathcal{A}$ of tree-width $\leq k$ satisfies ψ.

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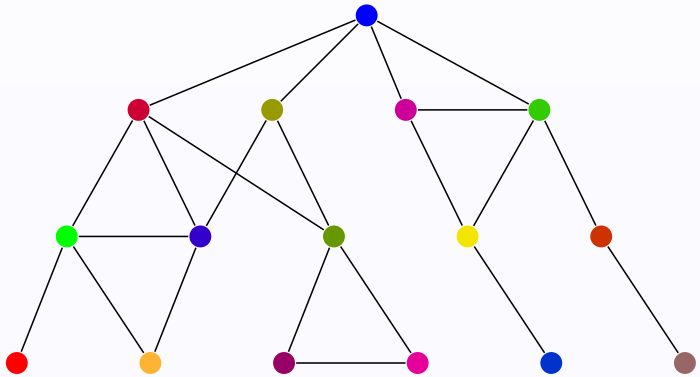
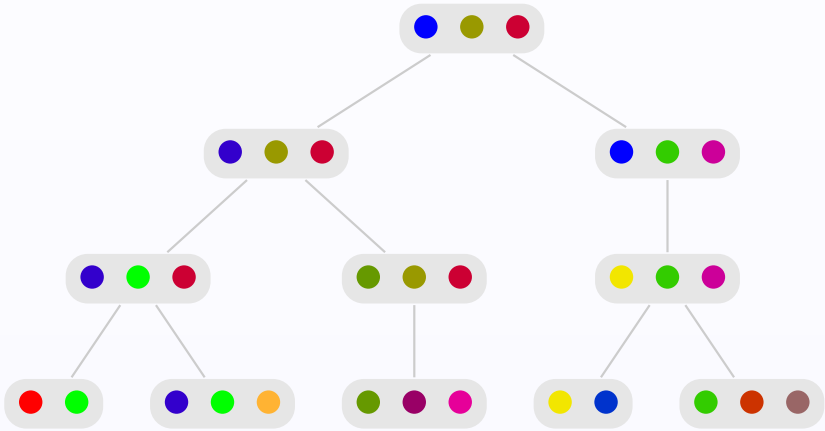


graph G



tree decomposition of width 2 for G

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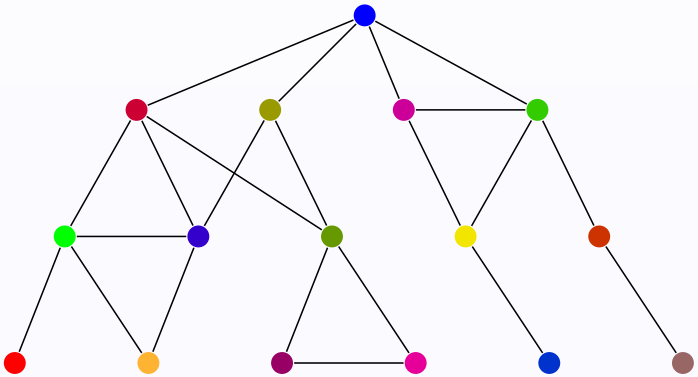

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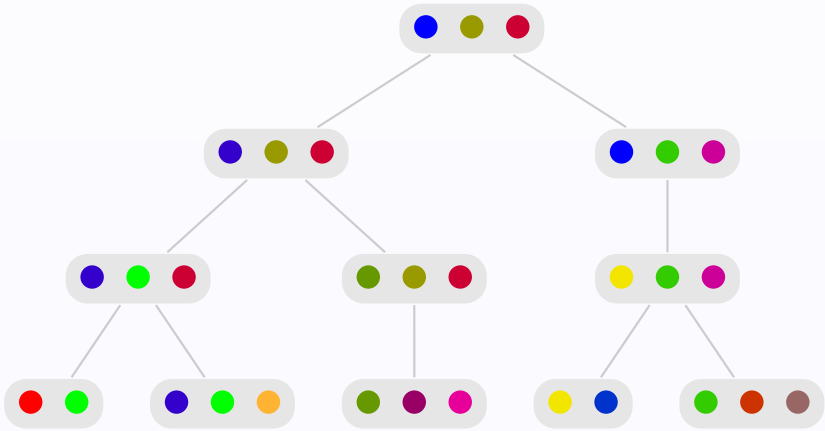

tree decomposition of width 2 for G

- Every vertex and every edge is contained in some bag
- For all vertices v: bags containing v form a subtree

Below polynomial time

Theorem [Elberfeld et al., 2010]

On structures of bounded tree-width, MSO-properties can be decided in logarithmic space.

Below polynomial time

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On structures of bounded tree-width, MSO-properties can be decided in logarithmic space.

$\longrightarrow$ Determine the exact (parallel/space) complexity of problems on bounded tree-width structures.

$$\mathbf{AC}^0 \subseteq \mathbf{TC}^0 \subseteq \mathbf{NC}^1 \subseteq \mathbf{L} \subseteq \mathbf{NC} \subseteq \mathbf{P} \subseteq \mathbf{NP}$$

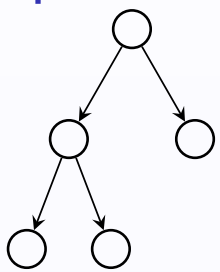
parallel alg. | sequential alg. | infeasible

Below logarithmic space

Theorem [Elberfeld et al., 2012]

On structures of bounded tree-width, MSO-properties can be decided in $\mathbf{NC}^1$, if tree decomposition is given as a term.

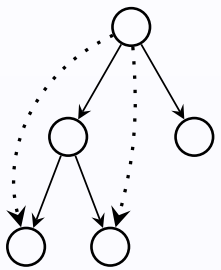
Tree representations



Pointer representation

(((())())())

Term representation $\equiv_{\mathbf{TC}^0}$



Ancestor representation

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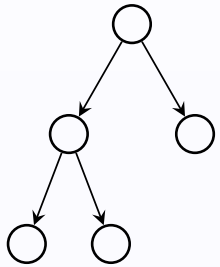

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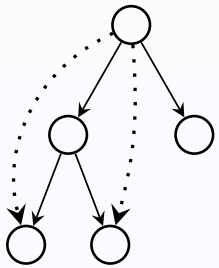


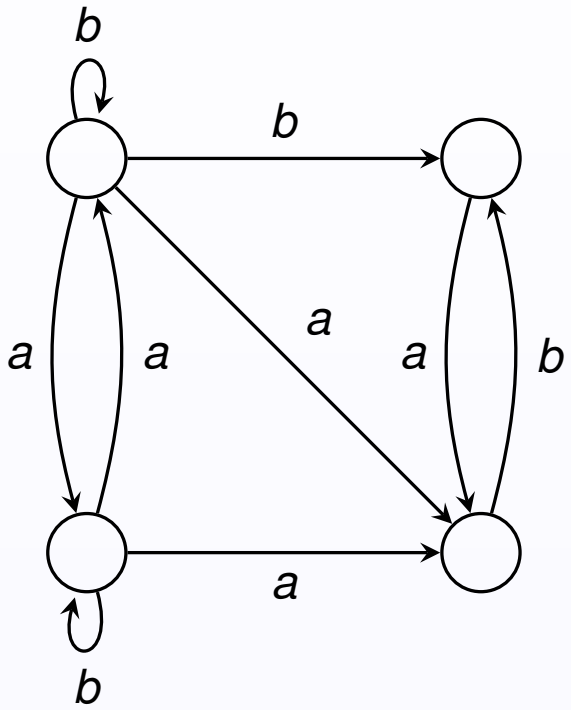
Pointer representation Term representation $\equiv_{\mathbf{TC}^0}$ Ancestor representation

Tree isomorphism problem

- In pointer representation: **L**-complete [Lindell, 1992]
- In term representation: $\mathbf{NC}^1$-complete [Buss, 1997]

Bisimulation game

Attacker and Defender alternatingly move two pebbles along transitions.



- If a player cannot move, the other player wins.
- Infinite plays are won by Defender.
- u, v are bisimilar if Defender has a winning strategy from (u, v).

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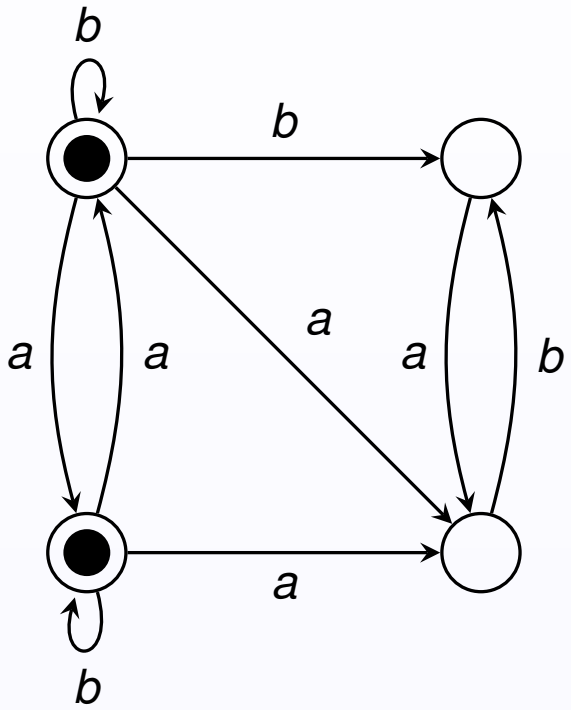

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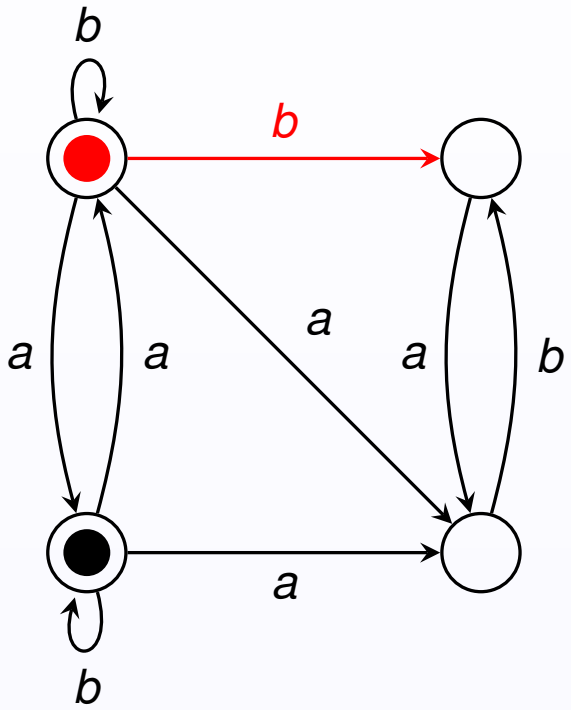

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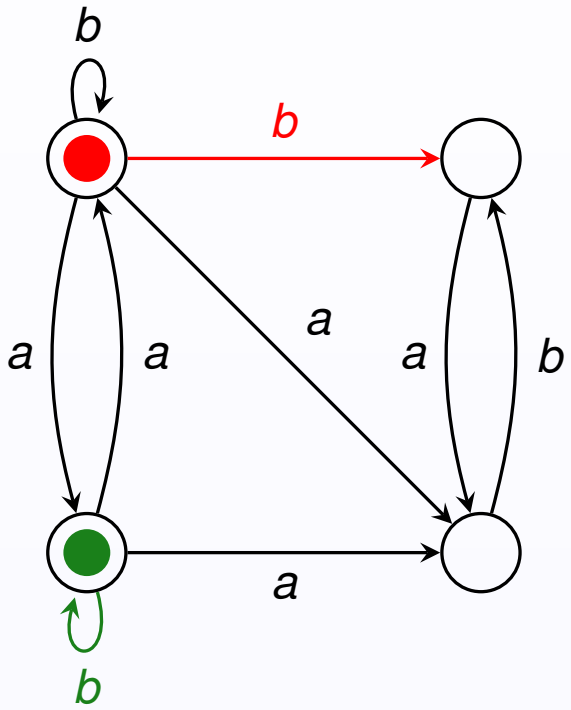

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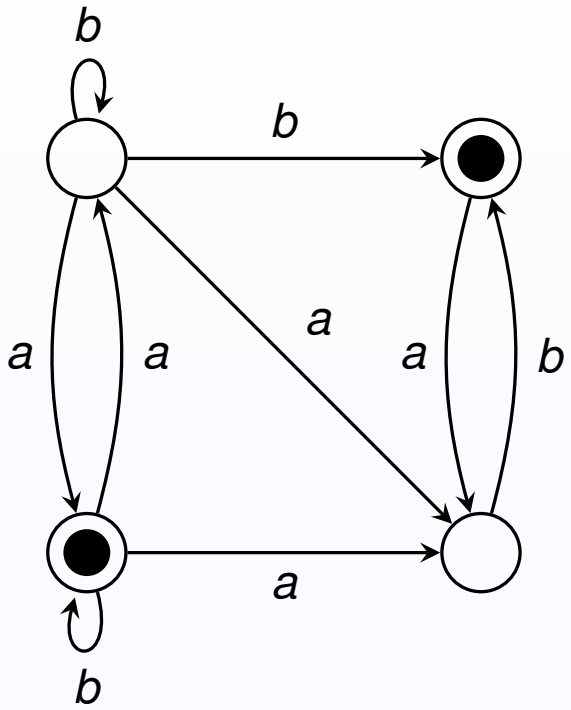

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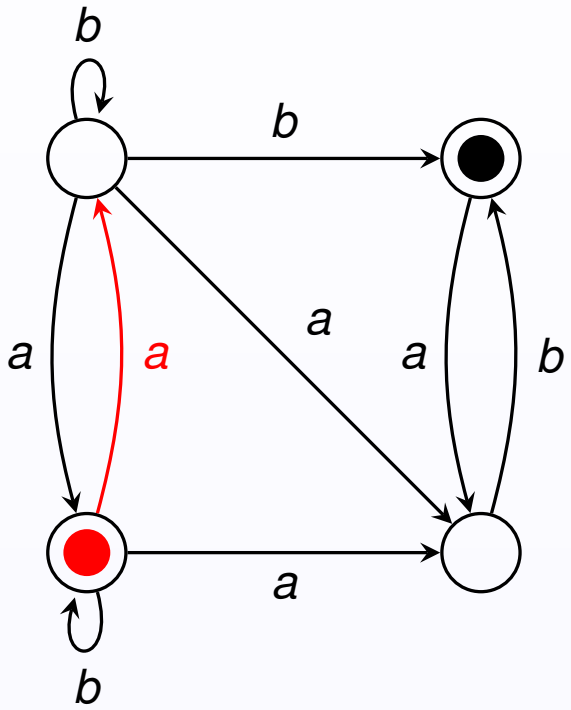

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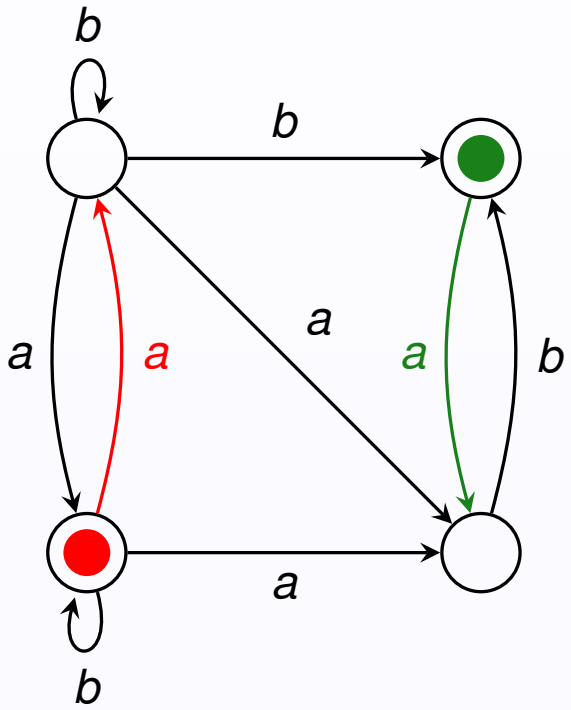

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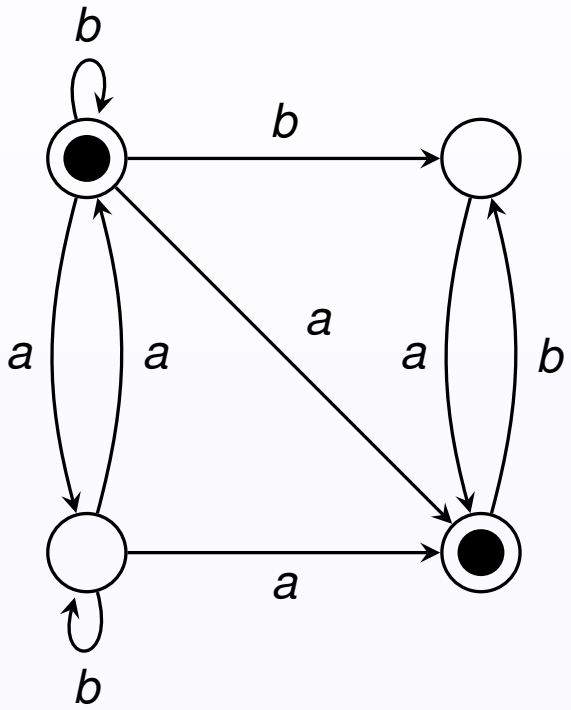

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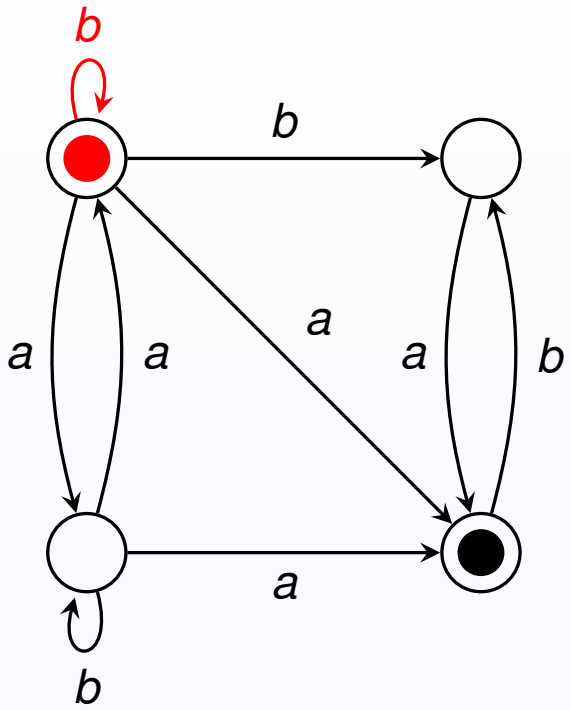

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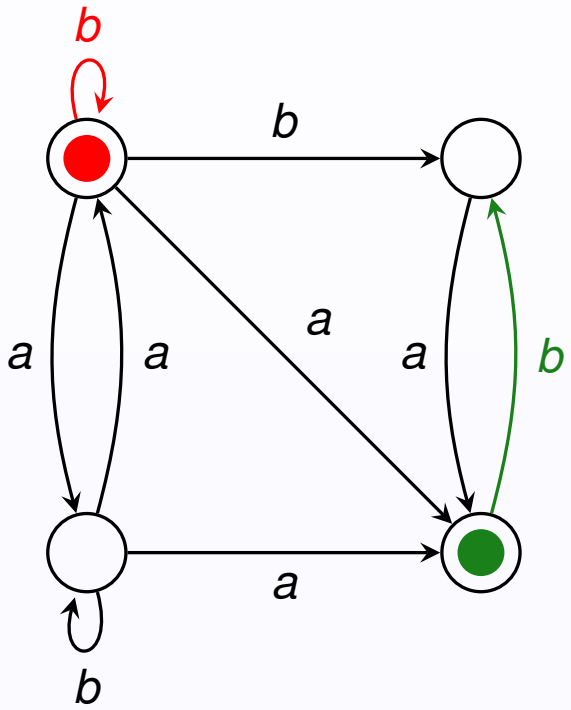

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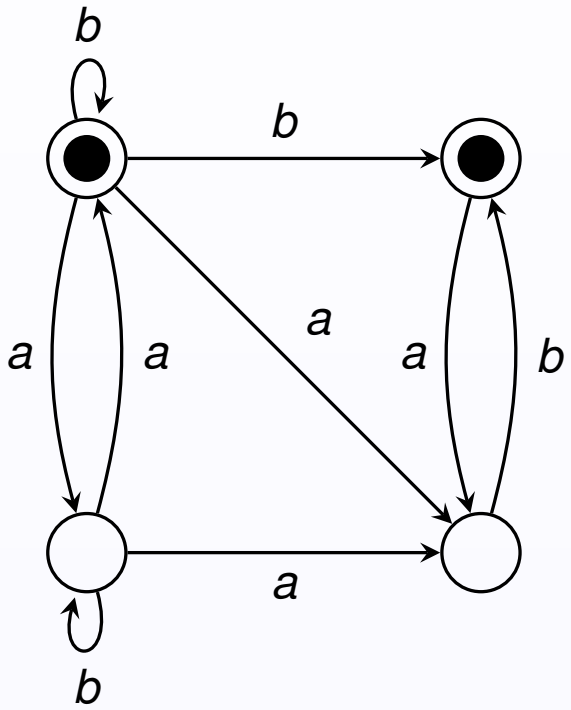

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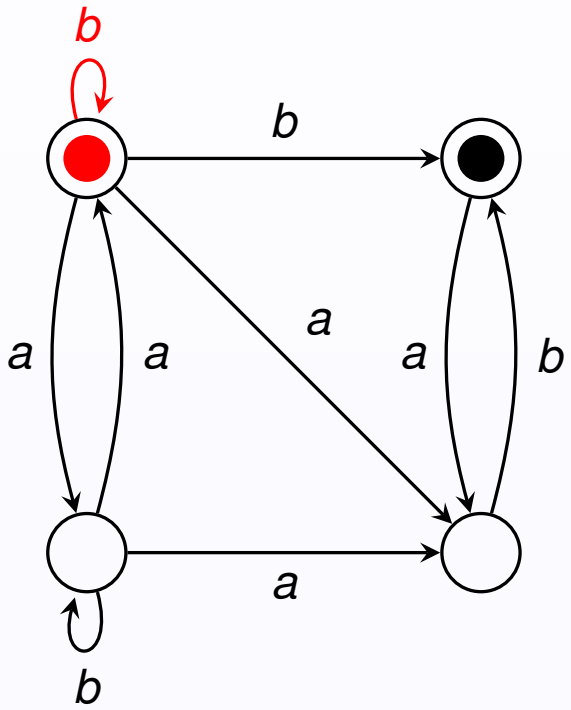

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Parallel complexity of bisimulation

- over finite systems: deciding bisimilarity is in **P** (partition refinement algorithm)
- infinite state systems: many (un)decidability results

Theorem [Balcazar et al., 1992]

The bisimulation problem over finite systems is **P**-complete.

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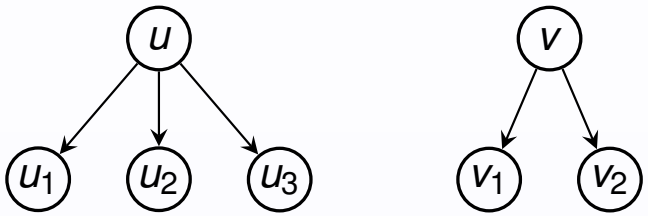
Theorem

The bisimulation problem for trees is

- in pointer representation: **L**-complete.
- in term representation: $\mathbf{NC}^1$-complete.

Bisimulation over unlabelled trees

Given two trees T_1, T_2, construct the bisimulation circuit $C(T_1, T_2)$:
gate $x_{u,v} = 1 \iff u$ and v are bisimilar



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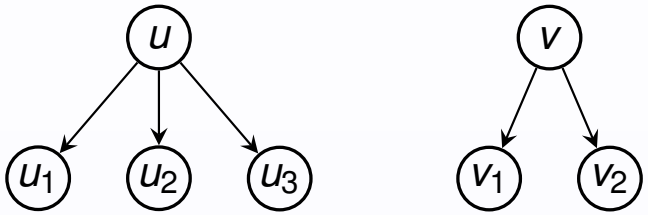

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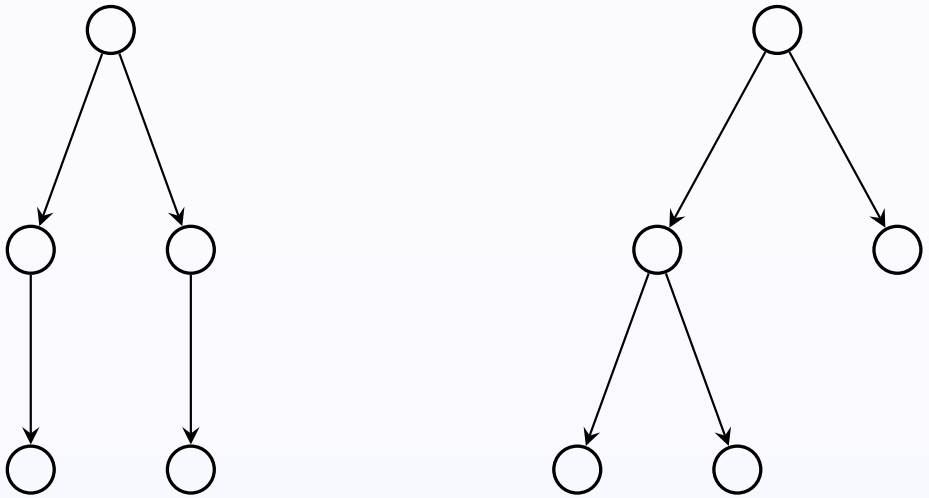

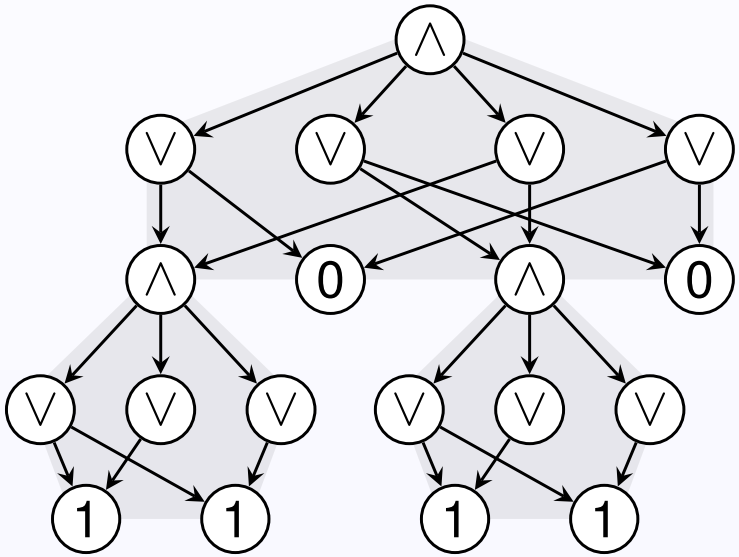

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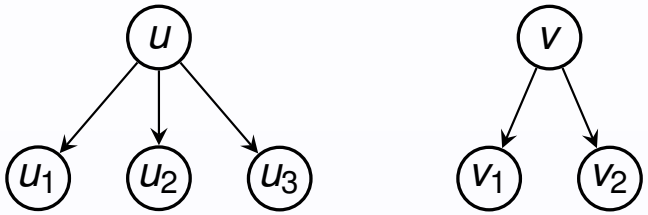

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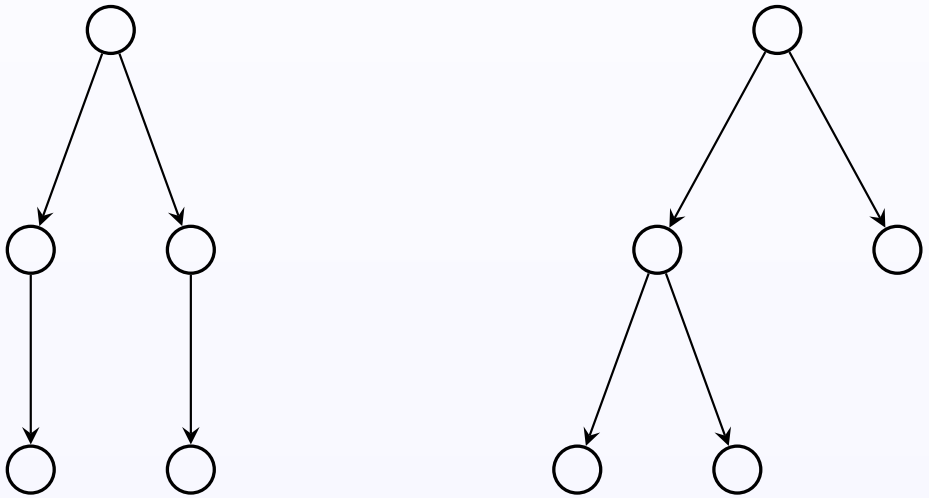
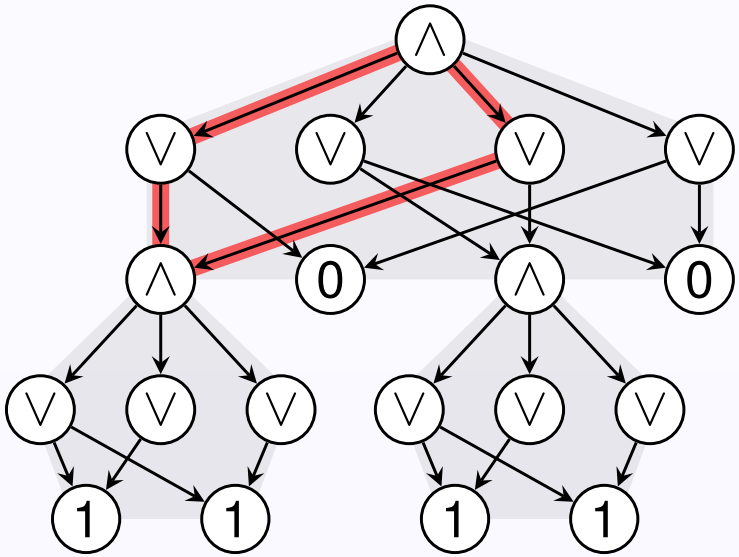
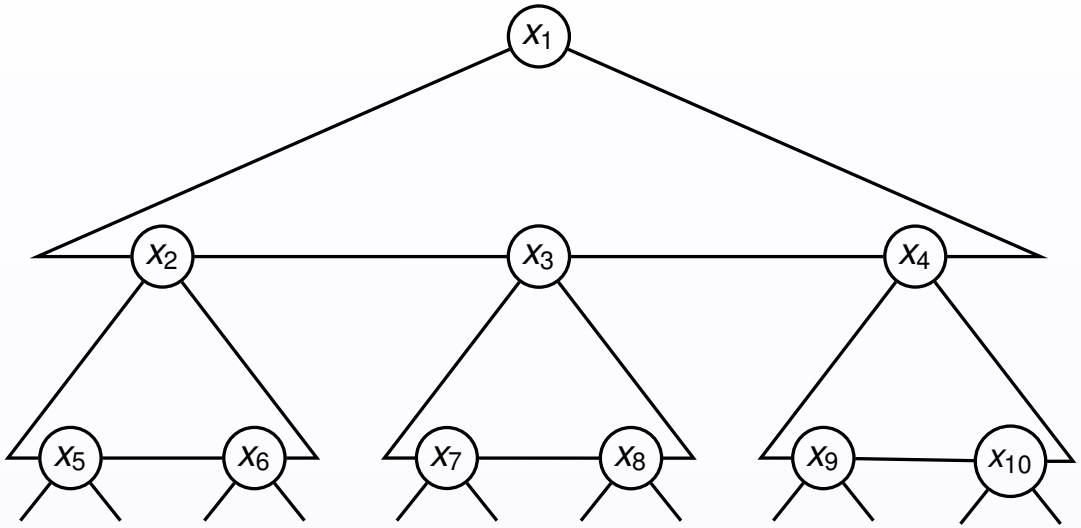




bounded number of paths

Tree-shaped circuits



Syntactic definition

$\mathcal{S} = (x_i := \varphi_i)_{1 \leq i \leq n}$

- Boolean formulas $\varphi_1, \ldots, \varphi_n$ over $x_1, \ldots, x_n$

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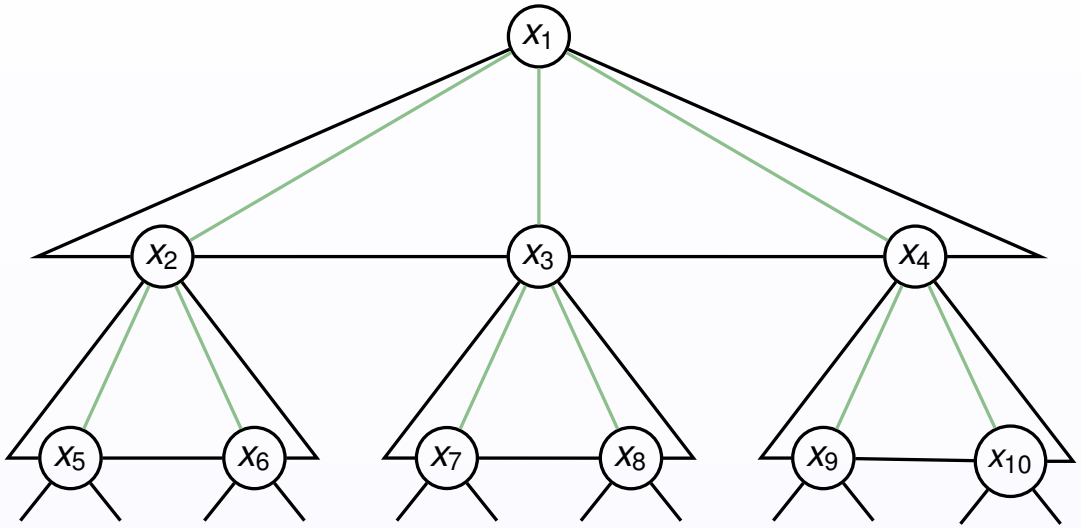

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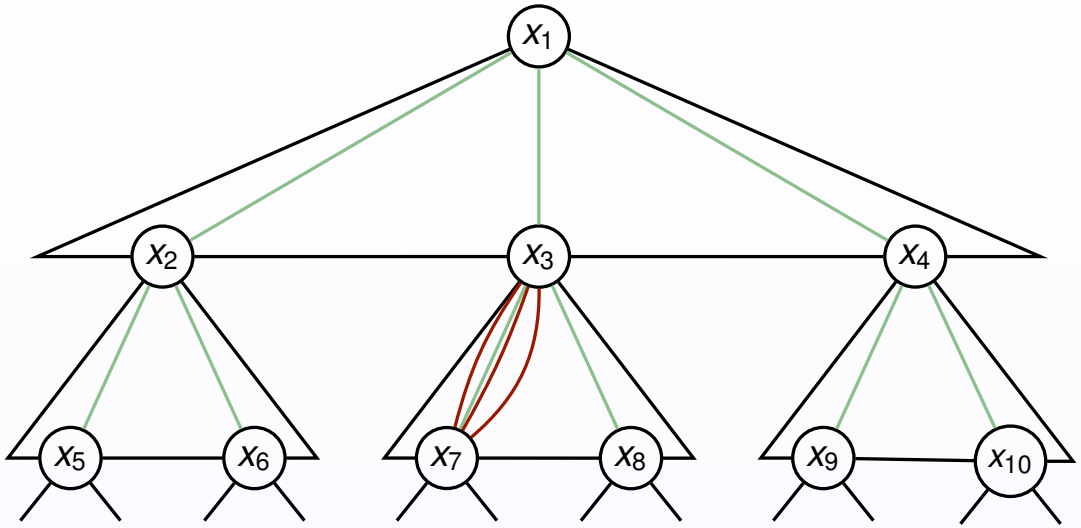

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- $T_{\mathcal{S}} = (\{x_1, \ldots, x_n\}, \{(x_i, x_j) : x_j \text{ occurs in } \varphi_i\})$ is a tree
- width of $\mathcal{S}$: max. number of occurrences of a variable x_j in some φ_i

Tree-shaped circuits

Theorem

For every $m \in \mathbb{N}$, evaluating tree-shaped circuits $\mathcal{S}$ of width $\leq m$ is in

- logarithmic space.
- $\mathbf{NC}^1$ if $T_{\mathcal{S}}$ is given as a term.

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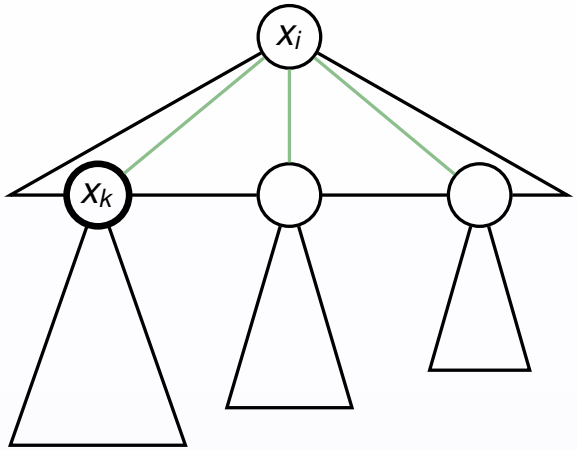
- Size of a tree node u: size of the subtree rooted in u
- u is heavy if $|u| \geq |v|$ for all siblings v of u, otherwise u is light.
- Each path in a tree T has $O(\log |T|)$ light nodes.

Preparation

Compute in $T_{\mathcal{S}}$ for each variable x_i its size and a heavy child x_k



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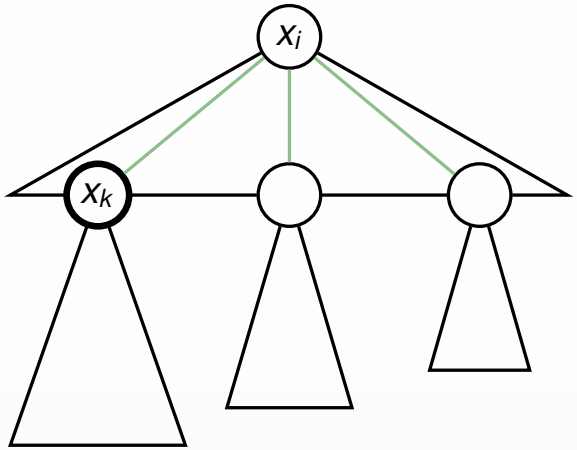

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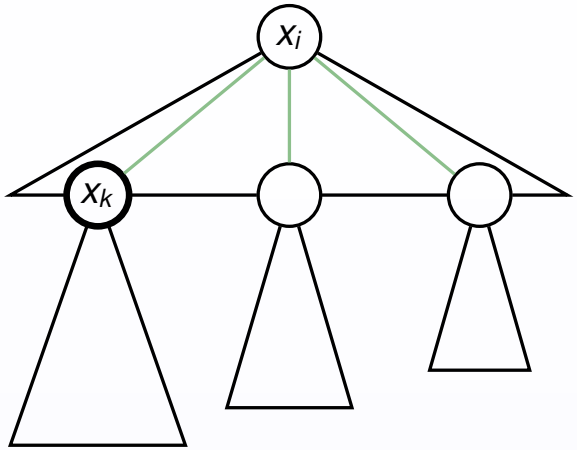

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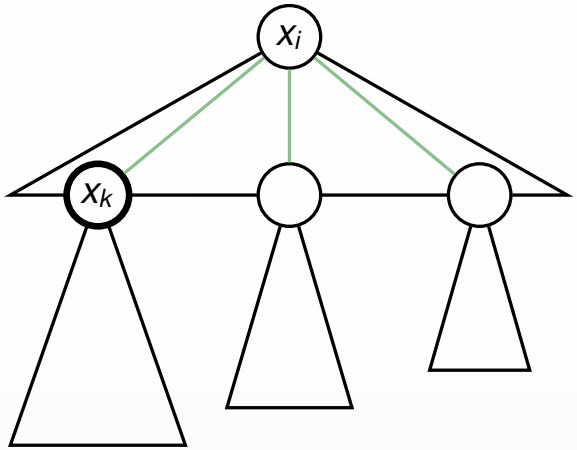

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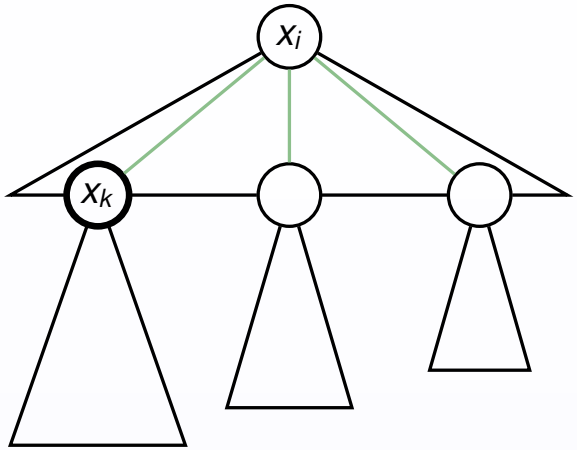

How to calculate x_i

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2. Evaluate φ_i by a depth-first traversal
   - value of x_k is known
   - each occurrence of a light variable $x_j \neq x_k$ is evaluated recursively (store position $d \in \{1, \ldots, m\}$ on the stack)

Alternative proof

Theorem

Circuits of bounded tree-width can be evaluated in logspace.
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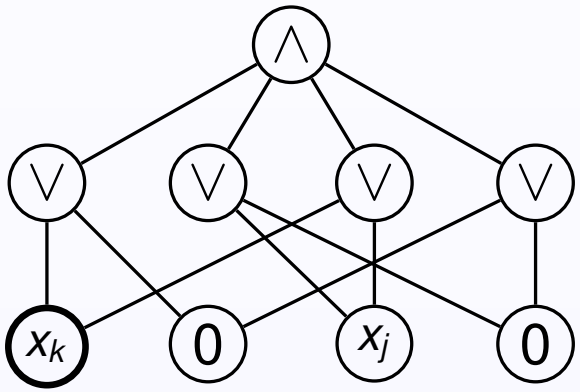
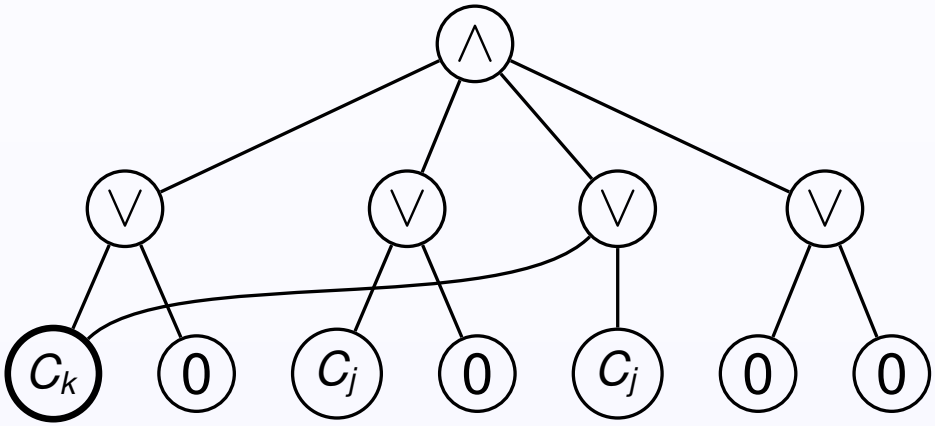
Partial unfolding



Subcircuit corresponding to φ_i



Partial unfolding C_i

Size of the partial unfolding

- $m = \text{width}(\mathcal{S})$.
- $\ell_i =$ number of light nodes from x_1 to x_i in $T_{\mathcal{S}}$.
- A gate x_i is copied at most m^{ℓ_i} times.

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Theorem

The set equality problem is $\mathbf{NC}^1$-complete.

From Trees to Bounded Tree-Width

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- Synchronized product creates grid-like structures
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Simulation problem

- Attacker (Defender) always moves the “left” (“right”) pebble
- u is simulated by v if Defender has a winning strategy

Simulation Problem

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The simulation problem on graphs of bounded path-width is **P**-hard.

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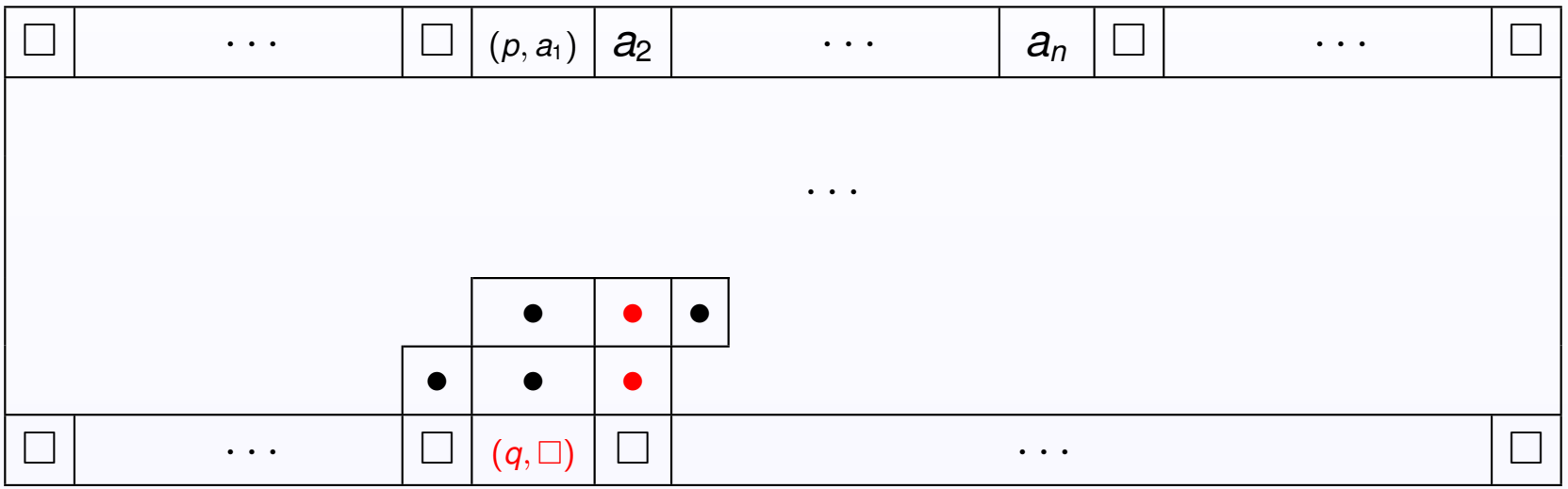
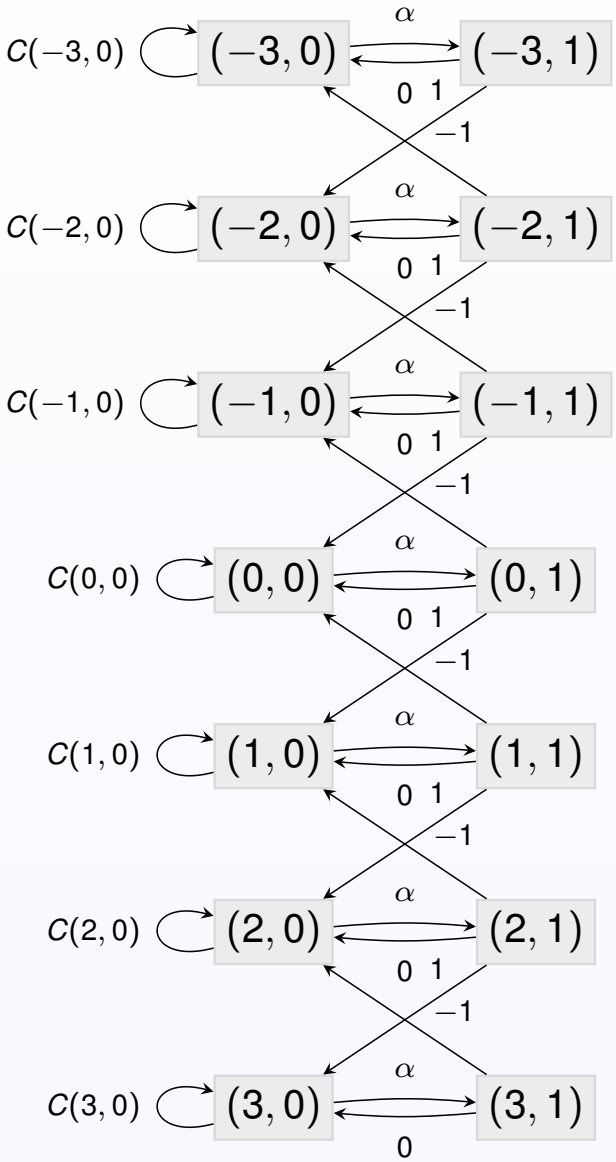
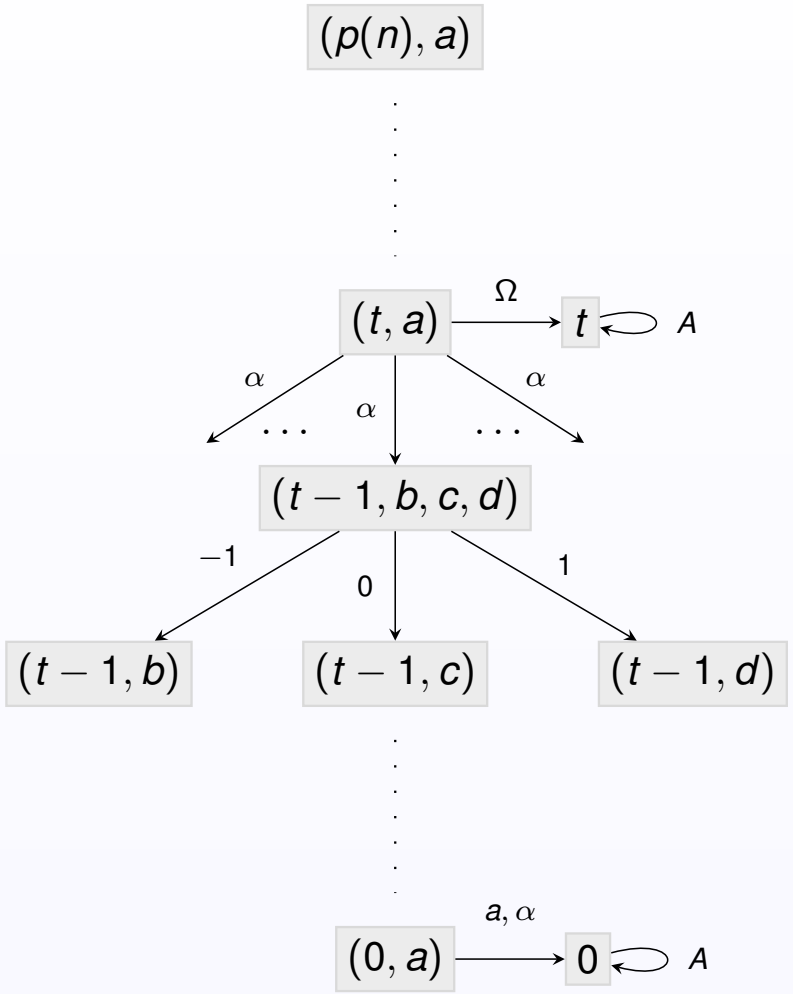
Fix a polynomial-time TM with a **P**-complete membership problem.



Proof sketch





Summary

The tree bisimulation problem is

- **L**-complete (for pointer structures)
- **NC**^{1}-complete (for term expressions)

Is there an **NC**-algorithm for bisimulation on graphs of bounded tree-width?