The directed homotopy hypothesis

CSL, Marseille

Jérémy Dubut (LSV, ENS Cachan, France)

Eric Goubault (LIX, Ecole Polytechnique, France)

Jean Goubault-Larrecq (LSV, ENS Cachan, France)

29th August, 2016

I.

Directed algebraic topology

Objective

Objective :

Compare spaces **with a notion of direction of time** up to continuous deformation **that preserves this direction**

Problem coming from :

- geometric semantics of truly concurrent systems
  - PV-programs **[Dijkstra 68]**
  - scan/update **[Afek et al. 90]**
  - higher dimensional automata **[Pratt 91]**
- theory of relativity **[Dodson, Poston 97]**

Non directed case : algebraic topology

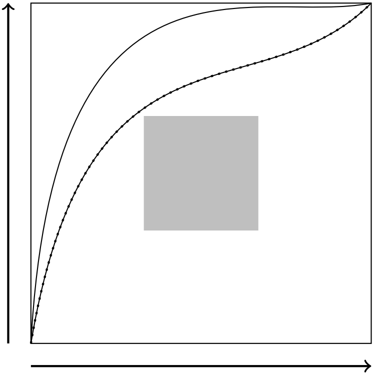
Non directed case : algebraic topology

Compare spaces ~~with a notion of direction of time~~ up to continuous deformation ~~that~~ ~~that preserves this direction~~

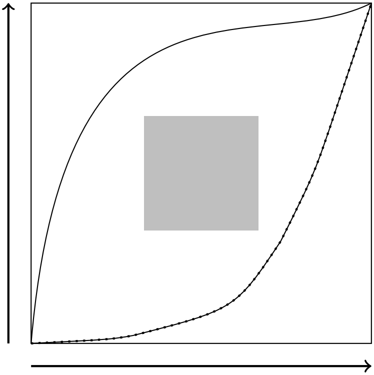
Dihomotopies

Directed space = topological space X with a collection of specified paths (continuous functions from $[0, 1]$ to X), called **di**paths

2 dipaths are **di**homotopic = you can deform continuously one into the other **while staying a dipath**

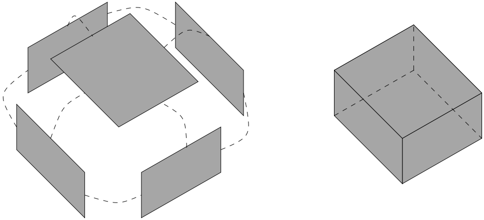


(di)homotopic



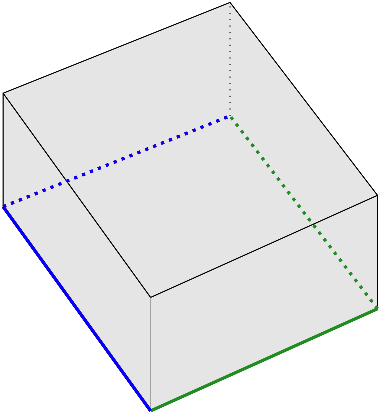
non (di)homotopic

Homotopy vs dihomotopy



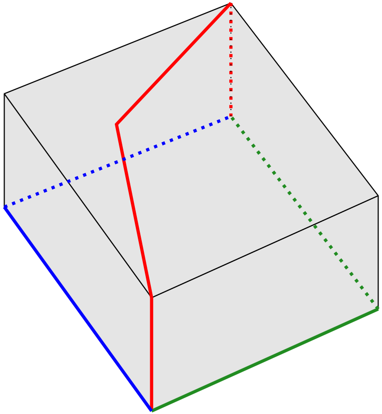
Fahrenberg's matchbox **[Fahrenberg 04]**

Homotopy vs dihomotopy



homotopic...

Homotopy vs dihomotopy



... but not dihomotopic

Purposes of our paper

- give algebraic representatives of directed spaces up to continuous deformation that preserves direction
- explicit what we mean by continuous deformation that preserves direction (through the notion of directed deformation retract)
- define a algebraic gadget (via a notion of "weak" enriched categories) that reflects directed phenomena

Theorem :

If two directed spaces are dihomotopy equivalent then their induced partially enriched categories are weakly equivalent.

II.

Grothendieck's homotopy hypothesis

Homotopy hypothesis : the motto

« Topological spaces are the same as ∞-groupoids. »

Topological spaces as ∞-groupoids

∞-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + ...

$$\begin{array}{rcl} \text{objects} & = & \text{points} \\ \text{1-cells} & = & \text{paths (= 0-homotopies)} \\ \text{2-cells} & = & \text{(1-)homotopies} \\ & \vdots & \\ \text{n-cells} & = & \text{(n-1)-homotopies} \end{array}$$

∞-groupoid = ∞-category whose n-cells are invertible up-to (n+1)-cells

Here : n-homotopies are invertible up-to (n+1)-homotopies

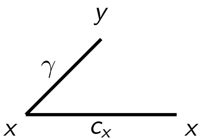
Ex : a path γ has $t \mapsto \gamma(1-t)$ as inverse up-to homotopy

But what are exactly ∞-groupoids ?

Many ways to « model » ∞-groupoids

$$\begin{array}{rcl} \infty\text{-groupoids} & = & \text{Kan complexes} \\ \text{n-cells} & = & \text{n-simplices} \\ \text{n-cells have inverse up-to (n+1)-cells} & = & \text{n-horns have (n+1)-fillers} \end{array}$$

Singular simplicial complex $Sing : Top \longrightarrow Kan\ (\subseteq Simp)$

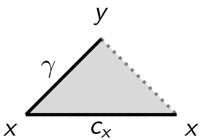


But what are exactly ∞-groupoids ?

Many ways to « model » ∞-groupoids

∞-groupoids	$=$	Kan complexes
n-cells	$=$	n-simplices
n-cells have inverse up-to (n+1)-cells	$=$	n-horns have (n+1)-fillers

Singular simplicial complex $\mathit{Sing} : \mathit{Top} \longrightarrow \mathit{Kan}$ $(\subseteq \mathit{Simp})$



A formal statement of the homotopy hypothesis

Theorem [Quillen 67] :

The Quillen-Serre model structure on topological spaces is Quillen-equivalent to the Kan-Quillen model structure on simplicial sets.

A few consequences :

- a topological space is weakly homotopy equivalent to the geometric realization of its singular simplicial complex (and so to a CW-complex)
- two topological spaces are weakly homotopy equivalent iff the geometric realization of their singular simplicial complex are weakly homotopy equivalent

A formal statement of the homotopy hypothesis

Theorem [Quillen 67] :

The Quillen-Serre model structure on topological spaces is Quillen-equivalent to the Kan-Quillen model structure on simplicial sets.

A few consequences :

- a topological space is weakly homotopy equivalent to the geometric realization of its singular simplicial complex (and so to a CW-complex)
- two topological spaces are weakly homotopy equivalent iff the geometric realization of their singular simplicial complex are weakly homotopy equivalent

« If two topological spaces are equivalent up-continuous deformation then their induced ∞-groupoids are equivalent (up-to weak equivalence in the suitable model structure) »

III.

A first proposal of directed homotopy hypothesis

Topological spaces as ∞-groupoids

∞-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + ...

$$\begin{array}{rcl} \text{objects} & = & \text{points} \\ \text{1-cells} & = & \text{paths (= 0-homotopies)} \\ \text{2-cells} & = & \text{(1-)homotopies} \\ & \vdots & \\ \text{n-cells} & = & \text{(n-1)-homotopies} \end{array}$$

∞-groupoid = ∞-category whose n-cells are invertible up-to (n+1)-cells

Here : n-homotopies are invertible up-to (n+1)-homotopies

Ex : a path γ has $t \mapsto \gamma(1-t)$ as inverse up-to homotopy

Directed topological spaces as ∞-groupoids

∞-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + ...

objects	=	points
1-cells	=	dipaths (= 0-dihomotopies)
2-cells	=	(1-)dihomotopies
	⋮	
n-cells	=	(n-1)-dihomotopies

∞-groupoid = ∞-category whose n-cells are invertible up-to (n+1)-cells

Here : n-dihomotopies are invertible up-to (n+1)-dihomotopies

Directed topological spaces as ∞-groupoids

∞-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + ...

objects	=	points
1-cells	=	dipaths (= 0-dihomotopies)
2-cells	=	(1-)dihomotopies
	⋮	
n-cells	=	(n-1)-dihomotopies

∞-groupoid = ∞-category whose n-cells are invertible up-to (n+1)-cells

Here : n-dihomotopies are invertible up-to (n+1)-dihomotopies

True for $n \geq 1$, but dipaths are not invertible up-to dihomotopy !

Directed topological spaces as $(\infty,1)$-categories

∞-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + ...

objects	=	points
1-cells	=	dipaths (= 0-dihomotopies)
2-cells	=	(1-)dihomotopies
	⋮	
n-cells	=	(n-1)-dihomotopies

$(\infty,1)$-category = ∞-category whose n-cells are invertible up-to (n+1)-cells for $n \geq 1$

Here : n-dihomotopies are invertible up-to (n+1)-dihomotopies for $n \geq 1$

Directed homotopy hypothesis : the motto ?

« Directed topological spaces are the same as $(\infty,1)$-categories. »

But what are exactly $(\infty, 1)$-categories ?

Many ways to « model » $(\infty, 1)$-categories :

- quasi-categories (= weak Kan complexes) **[Joyal]**
- enriched categories in Kan complexes **[Bergner]**
- ...

$(\infty, 1)$-categories	=	enriched categories in Kan complexes
objects	=	objects
n-cells	=	(n-1)-simplices of Hom-objects
n-cells have inverse up-to (n+1)-cells for $n \geq 1$	=	(n-1)-horns of Hom-objects have n-fillers for $n \geq 1$

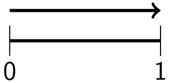
One direction of a directed homotopy hypothesis ?

Singular trace category $\mathbb{T} : dTop \longrightarrow KanCat \subseteq SimpCat$ **[Porter]**
$\mathbb{T}(X)$ = simplicially enriched category such that :

- objects = points of X
- Hom-object from x to y = singular simplicial complex of $\overrightarrow{T}(X)(x, y)$ (space of dipaths from x to y up-to increasing reparametrization)

« Can we compare (weak) dihomotopy types of directed spaces by their singular trace categories (up-to weak equivalence) ? »

Yes and no : the case of the directed segment



$\overrightarrow{I}$

In many equivalences, $\overrightarrow{I}$ is equivalent to a point $*$

$\mathbb{T}(\overrightarrow{I})$ and $\mathbb{T}(*)$ are not weakly equivalent.

Two problems

- Specify what we mean by equivalence directed spaces up-to continuous deformations which preserves directedness.
  - the match box not equivalent to a point
  - the directed segment equivalent to a point
  - few algebraic constructions are invariant (directed components **[Goubault, Haucourt 07]**, natural homology **[DGG 15]**)

- Fix the directed homotopy hypothesis.

III.

The need for equivalences in directed algebraic topology

Reminder on classical algebraic topology

A (strong) deformation retract of X on a subspace A is a continuous map

$$H : X \longrightarrow P(X) = [[0,1] \rightarrow X]$$

such that :

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0,1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$.

Theorem :

Two topological spaces are homotopy equivalent iff there is a span of deformation retracts between them.

Definition in directed algebraic topology

A future deformation retract of X on a sub-dspace A is a continuous map

$$H : X \longrightarrow \overrightarrow{P}(X)$$

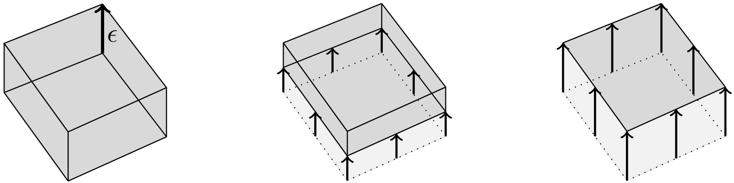
such that :

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0,1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$;
- for every $t \in [0,1]$, the map $H_t : x \mapsto H(x)(t)$ is a dmap ;
- for every δ of A from z to $H_1(x)$ there is a dipath γ of X from y to x with $H_1(y) = z$ and $H_1 \circ \gamma$ dihomotopic to δ.

Definition :

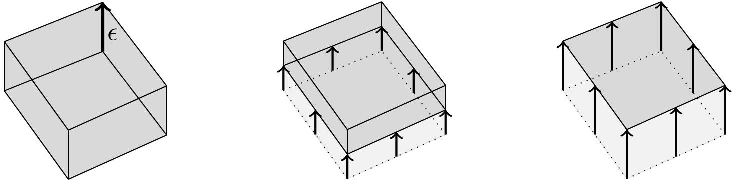
Two dspaces are dihomotopy equivalent iff there is a zigzag of future and past deformation retracts between them.

Something's wrong, isn't it ?



There is a future deformation retract from the matchbox to its upper face (and so to its upper corner) !

Something's wrong, isn't it ?



There is a future deformation retract from the matchbox to its upper face (and so to its upper corner) !

Problem : the dipaths along which we deform do not preserve the fact that dipaths are not dihomotopic.

Inessential dipaths

Idea from **[Fajstrup,Goubault, Haucourt,Raussen]** for category of components.

The set $\mathfrak{I}(X)$ of inessential dipaths of X is the largest set of dipaths such that :

- it is closed under concatenation and dihomotopy ;
- for every $\gamma \in \mathfrak{I}(X)$ from x to y, for every $z \in X$ such that $\overrightarrow{P}(X)(z,x)$, the map $\gamma \star \_ : \overrightarrow{P}(X)(z,x) \longrightarrow \overrightarrow{P}(X)(z,y) \;\; \delta \mapsto \gamma \star \delta$ is a homotopy equivalence ;
- symmetrically for $\_ \star \gamma$;
- $\mathfrak{I}(X)$ has the right and left Ore condition modulo dihomotopy :

$$\begin{array}{ccc} w & \overset{g'}{\dashrightarrow} & x \\ {\scriptstyle f' \in \mathfrak{I}(X)} \Big\downarrow & \text{mod. dihomot.} & \Big\downarrow {\scriptstyle f \in \mathfrak{I}(X)} \\ z & \underset{g}{\longrightarrow} & y \end{array} \qquad\qquad \begin{array}{ccc} z & \overset{g}{\longrightarrow} & y \\ {\scriptstyle f \in \mathfrak{I}(X)} \Big\downarrow & \text{mod. dihomot.} & \Big\downarrow {\scriptstyle f' \in \mathfrak{I}(X)} \\ x & \underset{g'}{\dashrightarrow} & w \end{array}$$

Ex : ϵ is not inessential in the matchbox

Better definition in directed algebraic topology

A future deformation retract of X on a sub-dspace A is a continuous map

$$H : X \longrightarrow \mathfrak{I}(X)$$

such that :

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0,1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$;
- for every $t \in [0,1]$, the map $H_t : x \mapsto H(x)(t)$ is a dmap ;
- for every δ of A from z to $H_1(x)$ there is a dipath γ of X from y to x with $H_1(y) = z$ and $H_1 \circ \gamma$ dihomotopic to δ.

Definition :

Two dspaces are dihomotopy equivalent iff there is a zigzag of future and past deformation retracts between them.

First results

- the directed segment is dihomotopy equivalent to a point
- the matchbox is not not dihomotopy equivalent to a point
- if two dspaces are dihomotopy equivalent then they have the same directed components and their natural homology are bisimilar

IV.

A new proposal of directed homotopy hypothesis

Fixation of the directed homotopy hypothesis

- replacing enriched categories by partially enriched categories (which encode accessibility)
- changing weak equivalences
- proving the following :

Theorem :

If two directed spaces are dihomotopy equivalent then their induced partially enriched categories are weakly equivalent.

Fixation of the directed homotopy hypothesis

- replacing enriched categories by partially enriched categories (which encode accessibility)
- changing weak equivalences
- proving the following :

Theorem :

If two directed spaces are dihomotopy equivalent then their induced partially enriched categories are weakly equivalent.

« One can compare directed spaces by comparing their partially enriched category (up-to weak equivalence). »

Conclusion

Summary :

- We have defined a dihomotopy equivalence, which behaves well on examples and for which natural homology is an invariant.
- We have defined a new structure, closed to $(\infty, 1)$-categories, and designed its weak equivalence, for which it is an invariant of dihomotopy equivalence.

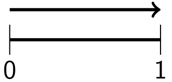
Many open questions :

- Are there two weakly equivalent dspaces that are not dihomotopy equivalent ?
- Are there model structures on dspaces (or partially enriched categories) for which the weak equivalence is dihomotopy equivalence (or weak equivalence) ?
- Do we have a kind of geometric realization from partially enriched categories to dspaces in order to formulate a complete directed homotopy equivalence ?
- Are the partially enriched categories (in Top or Simp) a nice model of $(\infty, 1)$-categories ?

IV.

A new proposal of directed homotopy hypothesis

The symptomatic case of the directed segment



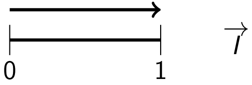
$\overrightarrow{I}$

In any reasonable equivalence, $\overrightarrow{I}$ is equivalent to a point $*$

$\mathbb{T}(\overrightarrow{I})$ and $\mathbb{T}(*)$ are not weakly equivalent :

- for $x < y$, $\overrightarrow{T}(\overrightarrow{I})(y, x)$ is empty while $\overrightarrow{T}(*)(*, *)$ is not
- their category of components are not equivalent (one has empty Hom-sets while the other has not)

The symptomatic case of the directed segment



In any reasonable equivalence, $\overrightarrow{I}$ is equivalent to a point $*$

$\mathbb{T}(\overrightarrow{I})$ and $\mathbb{T}(*)$ are not weakly equivalent :

- for $x < y$, $\overrightarrow{T}(\overrightarrow{I})(y, x)$ is empty while $\overrightarrow{T}(*)(*, *)$ is not
- their category of components are not equivalent (one has empty Hom-sets while the other has not)

Empty path spaces have a particular behavior that must be studied with care

Reminder on enriched categories and functors

Let $(V, U, \otimes)$ be a monoidal category.
A (small) enriched category $\mathcal{C}$ on V consists in the following data :

- a set of objects $Ob(\mathcal{C})$
- for every pair of objects A, B, an object $\mathcal{C}(A, B)$ of V
- for every triple of objects A, B, C, a morphism in V

$$\circ_{A,B,C} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$$

- for every object A, a morphism in V

$$u_A : U \longrightarrow \mathcal{C}(A, A)$$

satisfying some coherence diagrams (associativity, unity).

An enriched functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ on V consists in the following data :

- a function $F : Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{D})$;
- for every pair of objects A, B of $\mathcal{C}$, a morphism in V

$$F_{A,B} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(F(A), F(B))$$

satisfying some coherence diagrams (composition, unity).

A better definition to handle emptiness

Let $(V, U, \otimes)$ be a monoidal category.
A (small) partially enriched category $\mathcal{C}$ on V consists in the following data :

- a preordered set of objects $Ob(\mathcal{C})$, $\leq$
- for every pair of objects $A \leq B$, an object $\mathcal{C}(A, B)$ of V
- for every triple of objects $A \leq B \leq C$, a morphism in V

$$\circ_{A,B,C} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$$

- for every object A, a morphism in V

$$u_A : U \longrightarrow \mathcal{C}(A, A)$$

satisfying some coherence diagrams (associativity, unity), compatible with $\leq$.

An enriched functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ on V consists in the following data :

- a monotonic function $F : Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{D})$;
- for every pair of objects $A \leq B$ of $\mathcal{C}$, a morphism in V

$$F_{A,B} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(F(A), F(B))$$

satisfying some coherence diagrams (composition, unity), compatible with $\leq$.

From dTop to PeCat(HoTop) : the dipath category

$\mathbb{P}(X)$ = partially enriched category on *HoTop* :

- objects = points of X ;
- $x \leq y$ iff $\overrightarrow{P}(X)(x,y) \neq \varnothing$;
- for $x \leq y$, $\mathbb{P}(X)(x,y) = \overrightarrow{P}(X)(x,y)$;
- composition = concatenation up-to homotopy ;
- unit = constant path.

We can have defined it with value in *HoSimp* or *Ab* by composing with singular simplicial complex or homology.

We recover the fundamental category $\pi_1(X)$ by composing with the connected components functor.

What about the category of components ?

For **[Bergner]**, it is just $\pi_1(X)$.

What about the category of components?

For **[Bergner]**, it is just $\pi_1(X)$.

No good for directed segment.
Already known since **[Fajstrup, Goubault, Haucourt, Raussen]**.

What about the category of components ?

For **[Bergner]**, it is just $\pi_1(X)$.

No good for directed segment.
Already known since **[Fajstrup, Goubault, Haucourt, Raussen]**.

We have to define a category of « directed » components.

Yoneda morphisms, category of directed components

A slight modification of **[Fajstrup, Goubault, Haucourt, Raussen]**

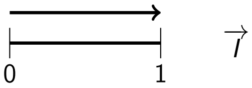
The set $\mathfrak{Y}(\mathcal{C})$ of Yoneda morphisms of a category $\mathcal{C}$ is the largest set of morphisms such that :

- it is closed under concatenation ;
- for every $f : c \longrightarrow c' \in \mathfrak{Y}(\mathcal{C})$, for every object c'' of $\mathcal{C}$ such that $\mathcal{C}(c', c'') \neq \emptyset$, the function $\_ \circ f : \mathcal{C}(c', c'') \longrightarrow \mathcal{C}(c, c'')$ $\; g \longmapsto g \circ f$ is a bijection ;
- symmetrically for $f \circ \_$;
- it has right and left Ore conditions

$$\begin{array}{ccc} w & \xrightarrow{g' \in \mathcal{C}} & x \\ {\scriptstyle f' \in \mathfrak{Y}(\mathcal{C})} \downarrow & & \downarrow {\scriptstyle f \in \mathfrak{Y}(\mathcal{C})} \\ z & \xrightarrow[g \in \mathcal{C}]{} & y \end{array} \qquad\qquad \begin{array}{ccc} z & \xrightarrow{g \in \mathcal{C}} & y \\ {\scriptstyle f \in \mathfrak{Y}(\mathcal{C})} \downarrow & & \downarrow {\scriptstyle f' \in \mathfrak{Y}(\mathcal{C})} \\ x & \xrightarrow[g' \in \mathcal{C}]{} & w \end{array}$$

$\overrightarrow{\pi_0}(\mathcal{C}) = \mathcal{C}[\mathfrak{Y}(\mathcal{C})^{-1}] = \mathcal{C}$ in which we inverse the morphisms in $\mathfrak{Y}(\mathcal{C})$
$\overrightarrow{\pi_0}(X) = \overrightarrow{\pi_0}(\pi_1(X))$

Example : the directed segment



$\mathbb{P}(\overrightarrow{I})$ is such that :

- $x \leq y$ is the usual ordering on I ;
- for every $x \leq y$, $\mathbb{P}(\overrightarrow{I})(x, y)$ is contractible.

The fundamental category $\pi_1(\overrightarrow{I})$ is the poset $(I, \leq)$.

The category of components $\pi_0(\overrightarrow{I})$ is the preordered set $(I, I \times I)$, which is equivalent to the category with one object and one morphism.

Weak dihomotopy equivalence

We say that a dmap $f : X \longrightarrow Y$ is a weak dihomotopy equivalence iff

- it induces an equivalence between the categories of directed components
- it induces a fully-faithful enriched functor between dipath categories i.e. for $x \leq_X x'$, the map

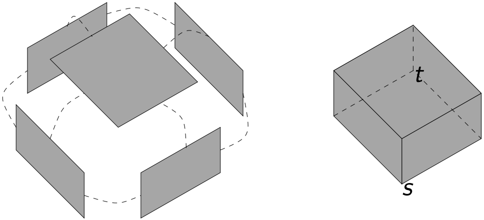
$$\mathbb{P}(f)_{x,x'} : \mathbb{P}(X)(x,x') \longrightarrow \mathbb{P}(Y)(f(x), f(x'))$$

  which maps γ to $f \circ \gamma$ is a homotopy equivalence.

We say that two dspaces are weakly dihomotopy equivalent iff there is zigzag of weak dihomotopy equivalence between them.

Examples

$\overrightarrow{I}$ is weakly equivalent to a point.



$\mathbb{P}(s, t)$ is homotopy equivalent to a two point space, so the match box cannot be weakly equivalent to a point.

Invariance

Theorem :

If two dspaces are dihomotopy equivalent, then they are weakly dihomotopy equivalent.

Invariance

Theorem :

If two dspaces are dihomotopy equivalent, then they are weakly dihomotopy equivalent.

« One can compare dspaces by comparing their dipath category (up-to weak equivalence). »

Invariance

Theorem :

If two dspaces are dihomotopy equivalent, then they are weakly dihomotopy equivalent.

« One can compare dspaces by comparing their dipath category (up-to weak equivalence). »

« Are dspaces the same as partially enriched categories in HoTop (or HoSimp) ? »