AC Dependency Pairs Revisited*

Akihisa Yamada, Christian Sternagel, René Thiemann
(University of Innsbruck, Austria)
Keiichirou Kusakari (Gifu University, Japan)

* Supported by FWF Y757 and P27502, and JSPS #24500012

Termination, AC termination

- Set of rewrite rules

$$\mathcal{R} = \begin{cases} 0 + y \to y \\ s(x) + y \to x + s(y) \end{cases}$$

is "terminating":

$$s(s(0)) + s(0) \underset{\mathcal{R}}{\to} s(0) + s(s(0)) \underset{\mathcal{R}}{\to} 0 + s(s(s(0))) \underset{\mathcal{R}}{\to} s(s(s(0)))$$

2 + 1   1 + 2   0 + 3   3

"DP framework" nicely proves it

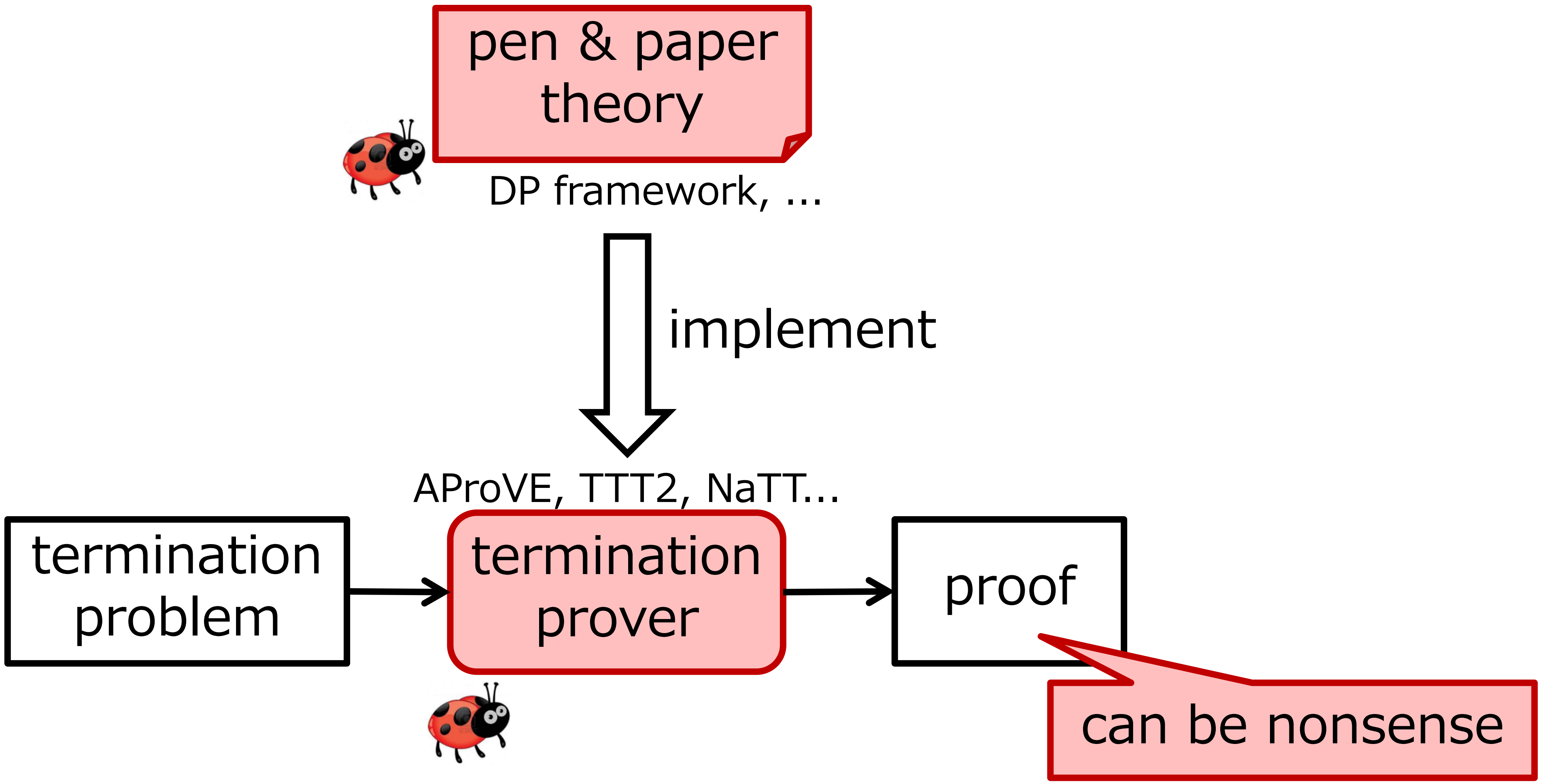
- but not "AC terminating":

$$s(s(0)) + s(0) \underset{\mathcal{R}}{\to} s(0) + s(s(0)) \approx s(s(0)) + s(0) \underset{\mathcal{R}}{\to} \cdots$$

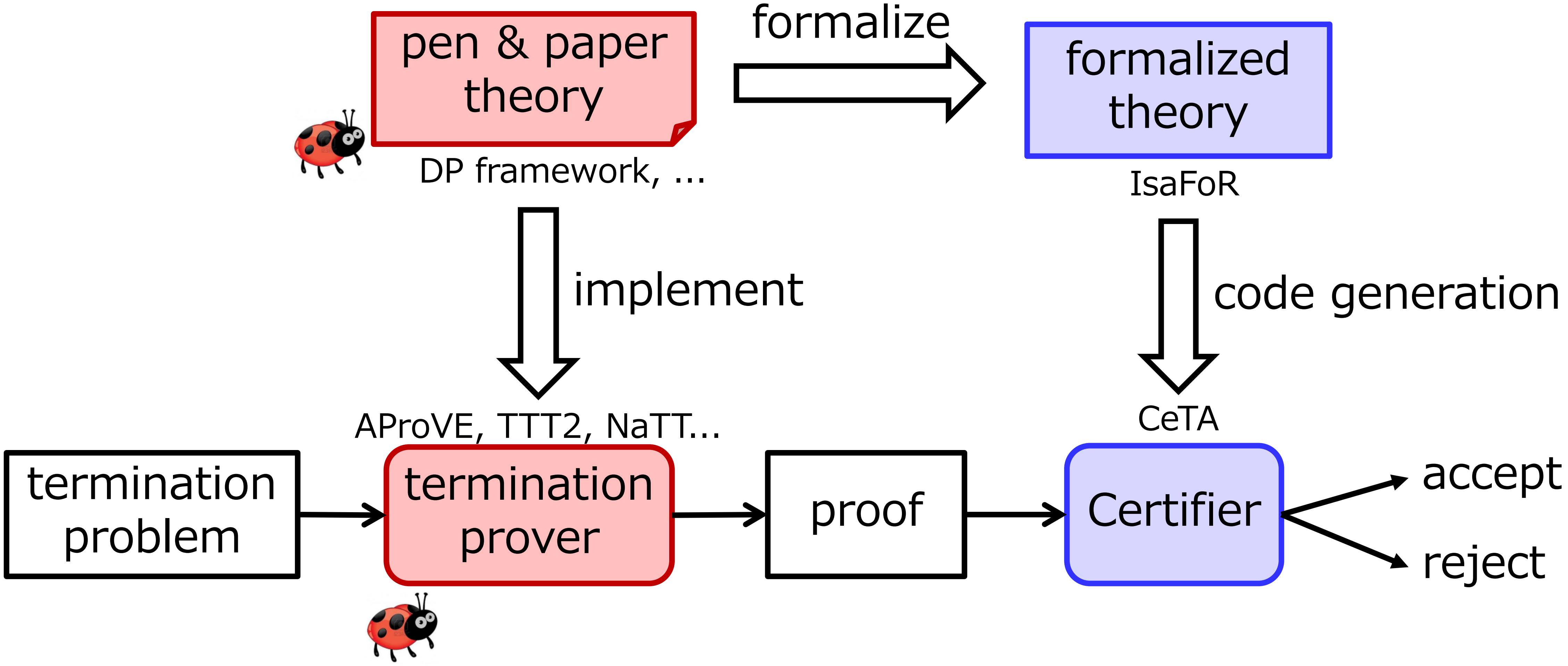
2 + 1   1 + 2   2 + 1

we provide a "certifiable" AC-DP framework

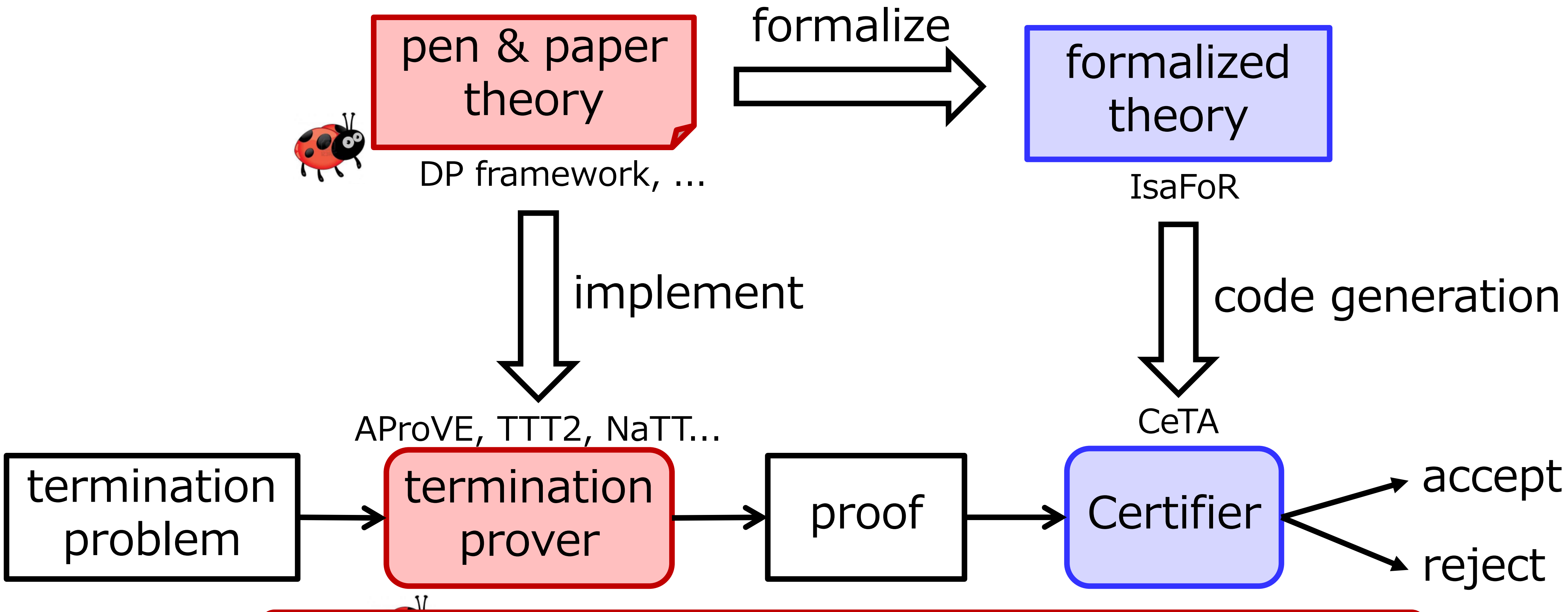
Big picture: Termination case



Big picture: Termination case

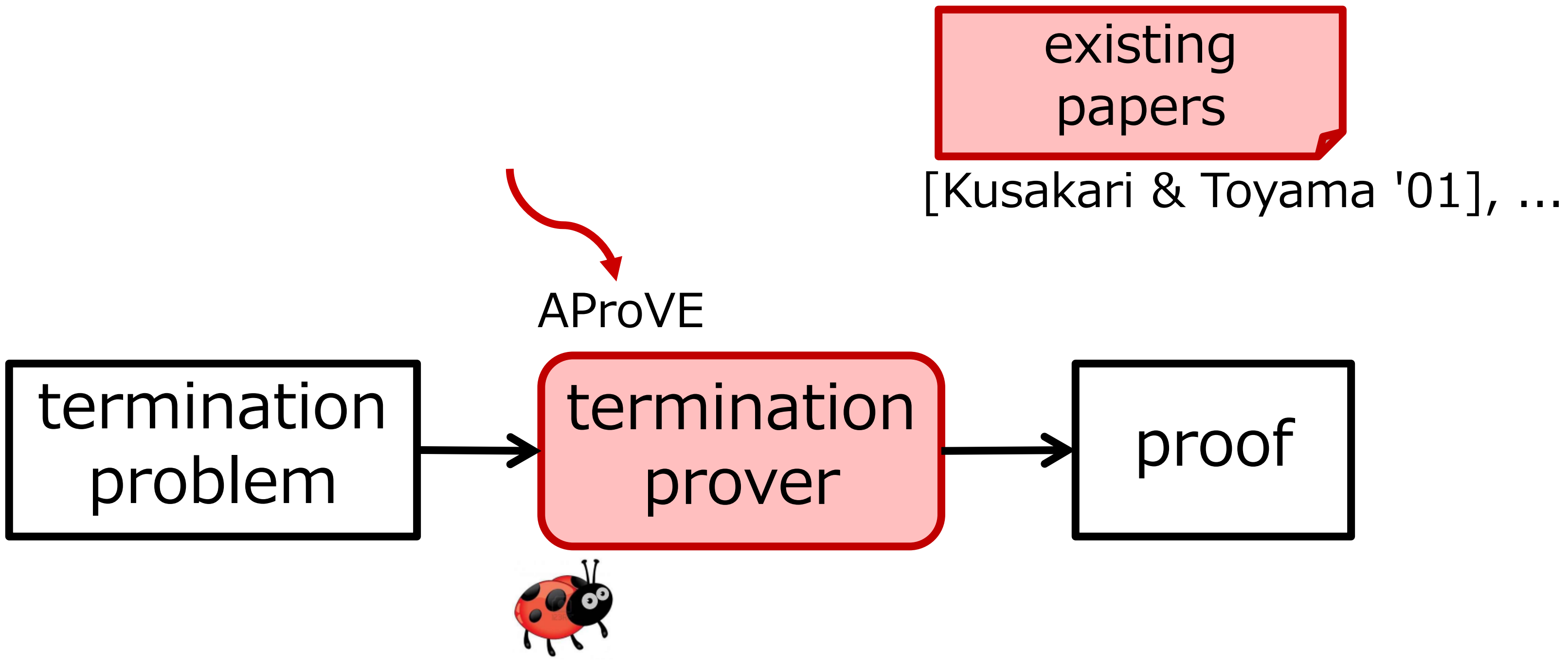


Big picture: Termination case

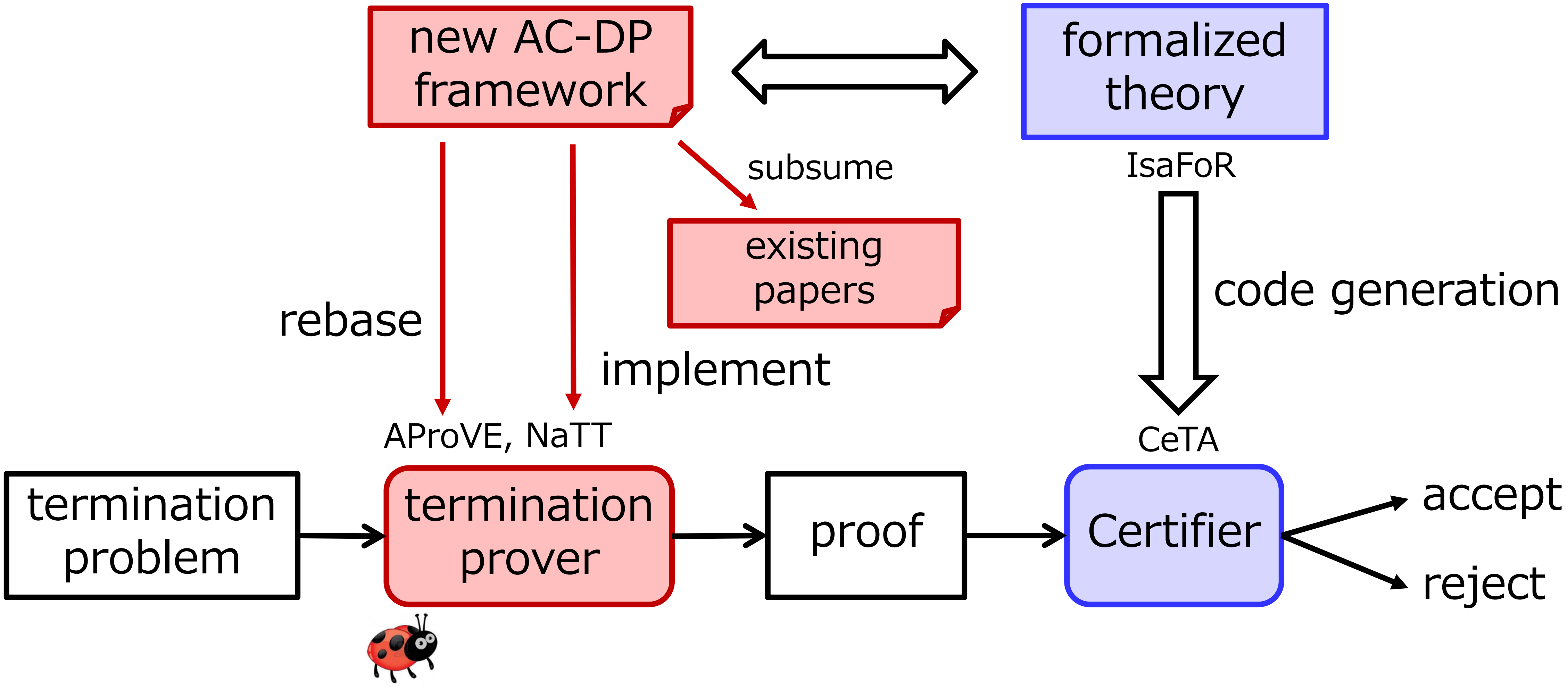


...for AC termination, it doesn't look so good...

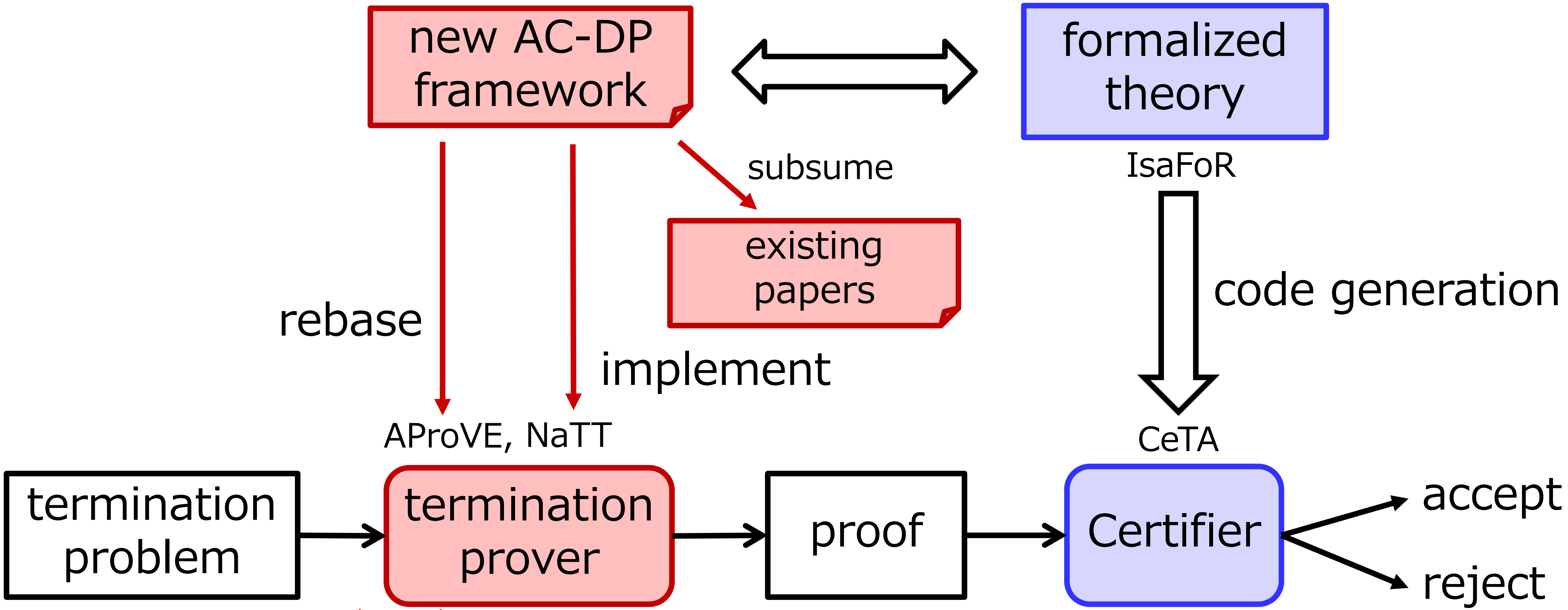
Big picture: AC termination case...



Big picture: Our work



Big picture: Our work



Found & fixed a bug in AProVE

Key idea

Dependency pairs (DP) [Arts & Giesl '00]

- Traces dependency in rewrite rules

$$\mathcal{R} = \begin{cases} 0 + y \to y \\ \mathrm{s}(x) + y \to \mathrm{s}(x + y) \end{cases}$$

makes NT terms terminating

$$\mathbf{DP}(\mathcal{R}) = \{\mathrm{s}(x) +^{\#} y \to x +^{\#} y\}$$

Theorem [Arts & Giesl '00]:
$\mathcal{R}$ is terminating ⇔ there exist no ⟨**DP(ℛ)**, ℛ⟩-"chain"

$$s_0^{\#} \xrightarrow[\mathbf{DP}(\mathcal{R})]{\epsilon} t_0^{\#} \to_{\mathcal{R}}^{*} s_1^{\#} \xrightarrow[\mathbf{DP}(\mathcal{R})]{\epsilon} t_1^{\#} \to_{\mathcal{R}}^{*} s_2^{\#} \xrightarrow[\mathbf{DP}(\mathcal{R})]{\epsilon} t_2^{\#} \to_{\mathcal{R}}^{*} \cdots$$

only count "root" steps

DP not for AC

- Not directly applicable

Example: [Kusakari & Toyama '01]

$$\mathcal{R} = \{\, x + x \to \mathrm{a} + \mathrm{b} \,\}$$

where + is associative.

$$\mathbf{DP(\mathcal{R})} = \{\, x +^{\#} x \to \mathrm{a} +^{\#} \mathrm{b} \,\}$$

cannot have infinite chain but

$$(\mathrm{a} + \mathrm{a}) + \mathrm{b} \underset{\mathcal{R}}{\to} (\mathrm{a} + \mathrm{b}) + \mathrm{b} \approx \mathrm{a} + (\mathrm{b} + \mathrm{b}) \underset{\mathcal{R}}{\to} \mathrm{a} + (\mathrm{a} + \mathrm{b}) \approx (\mathrm{a} + \mathrm{a}) + \mathrm{b} \underset{\mathcal{R}}{\to} \cdots$$

Our key idea: AC as rewrite rule

$$\mathcal{E} = \begin{cases} x + y \to y + x & \text{(C)} \\ x + (y + z) \to (x + y) + z & \text{(A)} \end{cases}$$

- Merit: DPs of $\mathcal{E}$

$$\mathrm{DP}(\mathcal{E}) = \begin{cases} x +^{\#} y \to y +^{\#} x & (\mathrm{C}^{\#}) \\ x +^{\#} (y + z) \to (x + y) +^{\#} z & (\mathrm{A}^{\#}) \\ x +^{\#} (y + z) \to x +^{\#} y & (\mathrm{a}^{\#}) \end{cases}$$

Still doesn't solve the problem, but
many non-AC results can be reused

Formal proof

inductive contradiction
(w.r.t. complicated order)

Formal proof

theorem

assumes …

shows $\mathcal{R}$ is AC terminating

Formal proof

theorem

assumes …

shows NT = ∅

terms that start infinite reduction

Formal proof

theorem

  assumes …

  and s ∈ NT **shows** False

Formal proof

theorem

assumes …

and $s^\# \in \text{minNT}$ **shows** False

$s = f(\ldots)$ is nonterminating but
$s^\# = f^\#(\ldots)$ is terminating

Formal proof

theorem

assumes …

and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof-

obtain $s \underset{\mathcal{E}}{\approx} \circ \underset{\mathcal{R}}{\rightarrow} \cdots$

Formal proof

theorem

assumes …

and $s^\# \in \mathrm{minNT}$ **shows** False

proof-

obtain $s^\# \approx \circ \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \underset{\mathcal{R}}{\rightarrow} \right) t^\# \in \mathrm{minNT}$

$\approx := \xrightarrow[\mathrm{DP}(\mathcal{E})]{\epsilon} \cup \underset{\mathcal{E}}{\rightarrow}$

Formal proof: case 1

theorem

assumes …

and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof-

obtain $s^{\#} \approx \circ \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \xrightarrow[\mathcal{R}]{} \right) t^{\#} \in \mathrm{minNT}$

Lemma: no $\langle \mathrm{DP}(\mathcal{R}) / \mathrm{DP}(\mathcal{E}), \mathcal{R}/\mathcal{E} \rangle$-chain $\Rightarrow$

$\succ_1 := \approx \circ \xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon}$ is well-founded

Formal proof: case 1

theorem

assumes no $\mathrm{DP}(\mathcal{R})$-chain

and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof-

obtain $s^{\#} \approx \circ \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \xrightarrow[\mathcal{R}]{} \right) \succ_1 t^{\#} \in \mathrm{minNT}$

Lemma: no $\langle \mathrm{DP}(\mathcal{R})/\mathrm{DP}(\mathcal{E}), \mathcal{R}/\mathcal{E}\rangle$-chain $\Rightarrow$

$\succ_1 := \approx \circ \xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon}$ is well-founded

Formal proof: case 1

theorem

assumes no $\mathrm{DP}(\mathcal{R})$-chain

and $s^\# \in \mathrm{minNT}$ **shows** False

proof(induct $s^\#$ w.r.t. $\succ_1$)

$s^\# \succ_1 t^\# \in \mathrm{minNT} \Rightarrow \mathrm{False}$

obtain $s^\# \approx \circ \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \xrightarrow[\mathcal{R}]{} \right) t^\# \in \mathrm{minNT}$ ($\succ_1$)

Lemma: no $\langle \mathrm{DP}(\mathcal{R})/\mathrm{DP}(\mathcal{E}), \mathcal{R}/\mathcal{E} \rangle$-chain $\Rightarrow$

$\succ_1 := \approx \circ \xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon}$ is well-founded

Formal proof: case 2

theorem

assumes no $\mathrm{DP}(\mathcal{R})$-chain

and $s^\# \in \mathrm{minNT}$ **shows** False

proof(induct $s^\#$ w.r.t. $\succ_1$)

obtain $s^\# \approx \circ \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \xrightarrow[\mathcal{R}]{} \right) t^\# \in \mathrm{minNT}$ ✓

Formal proof: case 2

theorem

assumes no $\mathrm{DP}(\mathcal{R})$-chain

and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof(induct $s^{\#}$ w.r.t. $\succ_1$)

obtain $s^{\#} \approx \circ \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \xrightarrow[\mathcal{R}]{\mathrm{top}} \cup \xrightarrow[\mathcal{R}]{>\mathrm{top}} \right) t^{\#} \in \mathrm{minNT}$

Definition $(\xrightarrow[\mathcal{R}]{\mathrm{top}})$:

$$s_1 + \cdots + \boxed{s_k + s_{k+1}} + \cdots + s_n \xrightarrow[\mathcal{R}]{\mathrm{top}} s_1 + \cdots + \boxed{t} + \cdots + s_n$$

Formal proof: case 2.1

theorem

assumes no $\mathrm{DP}(\mathcal{R})$-chain

and $s^\# \in \mathrm{minNT}$ **shows** False

proof(induct $s^\#$ w.r.t. $\succ_1$)

obtain $s^\# \approx \circ \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \xrightarrow[\mathcal{R}]{\mathrm{top}} \cup \xrightarrow[\mathcal{R}]{>\mathrm{top}} \right) t^\# \in \mathrm{minNT}$

Definition $(\xrightarrow[\mathcal{R}]{\mathrm{top}})$: if $l_1 + l_2 \to r \in \mathcal{R}$, then

$$\cdots + l_1\sigma + l_2\sigma + \cdots \xrightarrow[\mathcal{R}]{\mathrm{top}} \cdots + r\sigma + \cdots$$

AC-extended rewriting [Peterson & Stickel '81]

- $\mathcal{R}_\mathcal{E}$ contains for every

$$
\begin{array}{rcll}
l_1 + l_2 & \to & r & \in \mathcal{R} \\
x + l_1 + l_2 & \to & x + r & \text{if } + \text{ is associative, and further} \\
l_1 + l_2 + y & \to & r + y & \left.\right\} \text{if } + \text{ is not commutative} \\
x + l_1 + l_2 + y & \to & x + r + y &
\end{array}
$$

Lemma:

$$t \xrightarrow[\mathcal{R}]{\text{top}} w \quad \text{implies } t \approx u \xrightarrow[\mathcal{R}_\mathcal{E}]{\epsilon} v \approx w$$

AC-extended rewriting [Peterson & Stickel '81]

- $\mathcal{R}^{\#}_{\mathcal{E}}$ contains for every

$$
\begin{array}{rcll}
l_1 + l_2 & \to & r & \in \mathcal{R} \\
x +^{\#} l_1 + l_2 & \to & x +^{\#} r & \text{if } + \text{ is associative, and further} \\
l_1 + l_2 +^{\#} y & \to & r +^{\#} y & \\
x +^{\#} l_1 + l_2 + y & \to & x +^{\#} r + y & \text{if } + \text{ is not commutative}
\end{array}
$$

Lemma:

$$t^{\#} \xrightarrow[\mathcal{R}]{\text{top}} w^{\#} \text{ implies } t^{\#} \approx u^{\#} \xrightarrow[\mathcal{R}^{\#}_{\mathcal{E}}]{\epsilon} v^{\#} \approx w^{\#}$$

Formal proof: case 2.1

theorem

assumes no $\mathrm{DP}(\mathcal{R})$-chain

and $s^\# \in \mathrm{minNT}$ **shows** False

proof(induct $s^\#$ w.r.t. $\succ_1$)

obtain $s^\# \approx t^\# \approx u^\# \xrightarrow[\mathcal{R}_\mathcal{E}^\#]{\epsilon} v^\# \approx w^\# \in \mathrm{minNT}$

got a nicer sequence

now the same trick as in the previous case works!

Formal proof: case 2.1

theorem

assumes no $\mathrm{DP}(\mathcal{R})$-chain

and $s^\# \in \mathrm{minNT}$ **shows** False

proof(induct $s^\#$ w.r.t. $\succ_1$)

obtain $s^\# \approx t^\# \approx u^\# \xrightarrow[\mathcal{R}_\mathcal{E}^\#]{\epsilon} v^\# \approx w^\# \in \mathrm{minNT}$

Lemma: no $\mathcal{R}_\mathcal{E}^\#$-chain ⇒

$\succ_2 := \approx \circ \xrightarrow[\mathcal{R}_\mathcal{E}^\#]{\epsilon}$ is well-founded on minNT terms

Formal proof: case 2.1

theorem

assumes no $\mathrm{DP}(\mathcal{R})$- or $\mathcal{R}^{\#}_{\mathcal{E}}$-chain

and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof(induct $\langle s^{\#}, s^{\#}\rangle$ w.r.t. $\langle \succ_1, \succ_2 \rangle$)

$s^{\#} \succcurlyeq_1 v^{\#} \wedge$
$s^{\#} \succ_2 v^{\#} \wedge$
$v^{\#} \in \mathrm{minNT} \Rightarrow \mathrm{False}$

obtain $s^{\#} \approx t^{\#} \succ_2 u^{\#} \xrightarrow[\mathcal{R}^{\#}_{\mathcal{E}}]{\epsilon} v^{\#} (\approx w^{\#} \in \mathrm{minNT})$

Lemma: no $\mathcal{R}^{\#}_{\mathcal{E}}$-chain $\Rightarrow$

$\succ_2 := \approx \circ \xrightarrow[\mathcal{R}^{\#}_{\mathcal{E}}]{\epsilon}$ is well-founded on minNT terms ?

Formal proof: case 2.1.1

theorem

assumes no DP($\mathcal{R}$)- or $\mathcal{R}^{\#}_{\mathcal{E}}$-chain

and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof(induct $\langle s^{\#}, s^{\#}\rangle$ w.r.t. $\langle \succ_1, \succ_2 \rangle$)

$s^{\#} \succcurlyeq_1 v^{\#} \wedge$
$s^{\#} \succ_2 v^{\#} \wedge$
$v^{\#} \in \mathrm{minNT} \Rightarrow \mathrm{False}$

obtain $s^{\#} \approx t^{\#} \succ_2 u^{\#} \xrightarrow[\mathcal{R}^{\#}_{\mathcal{E}}]{\epsilon} v^{\#} \in \mathrm{minNT}$ ✓

Formal proof: case 2.1.2

theorem

assumes no $\mathrm{DP}(\mathcal{R})$- or $\mathcal{R}^{\#}_{\mathcal{E}}$-chain

and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof(induct $\langle s^{\#}, s^{\#} \rangle$ w.r.t. $\langle \succ_1, \succ_2 \rangle$)

obtain $s^{\#} \approx t^{\#} \approx u^{\#} \xrightarrow[\mathcal{R}^{\#}_{\mathcal{E}}]{\epsilon} v^{\#} \notin \mathrm{minNT}$

Formal proof: case 2.1.2

theorem

assumes no $\mathrm{DP}(\mathcal{R})$- or $\mathcal{R}_{\mathcal{E}}^{\#}$-chain

and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof(induct $\langle s^{\#}, s^{\#} \rangle$ w.r.t. $\langle \succ_1, \succ_2 \rangle$)

obtain $s^{\#} \approx t^{\#} \approx u^{\#} \xrightarrow[\mathcal{R}_{\mathcal{E}}^{\#}]{\epsilon} v^{\#} \notin \mathrm{minNT}$

$\rhd u'^{\#} \in \mathrm{minNT}$

$\nabla s \approx^{\mathrm{mul}} \nabla u \supset \nabla u'$

Definition (top flattening [Rubio '02]):

$\nabla(s_1 + s_2 + \cdots + s_n) := \{s_1, s_2, \ldots, s_n\}$ (multiset)

Formal proof

theorem

  assumes no $\mathrm{DP}(\mathcal{R})$- or $\mathcal{R}^{\#}_{\mathcal{E}}$-chain

  and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof(induct $\langle s^{\#}, s^{\#}, \nabla s \rangle$ w.r.t. $\langle \succ_1, \succ_2, \succ_3^{\mathrm{mul}} \rangle$)

  obtain $s^{\#} \approx t^{\#} \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \xrightarrow[\mathcal{R}]{\mathrm{top}} \cup \xrightarrow[\mathcal{R}]{>\mathrm{top}} \right) u^{\#} \in \mathrm{minNT}$ ✓ ✓

Formal proof: case 2.2

theorem

assumes no $\mathrm{DP}(\mathcal{R})$- or $\mathcal{R}^{\#}_{\mathcal{E}}$-chain

and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof(induct $\langle s^{\#}, s^{\#}, \nabla s\rangle$ w.r.t. $\langle \succ_1, \succ_2, \succ_3^{\mathrm{mul}}\rangle$)

obtain $s^{\#} \approx t^{\#} \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \xrightarrow[\mathcal{R}]{\mathrm{top}} \cup \xrightarrow[\mathcal{R}]{>\mathrm{top}}\right) u^{\#} \in \mathrm{minNT}$

$\nabla s \approx^{\mathrm{mul}} \nabla t \succ_3^{\mathrm{mul}} \nabla u$

Lemma: $\succ_3^{\mathrm{mul}}$ is well-founded (on minNT), where

$$s \succ_3 t \;\coloneqq\; s \approx \circ \xrightarrow[\mathcal{R}]{} \circ \approx t$$

Formal proof

theorem

  assumes no $\mathrm{DP}(\mathcal{R})$- or $\mathcal{R}_{\mathcal{E}}^{\#}$-chain

  and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof(induct $\langle s^{\#}, s^{\#}, \nabla s \rangle$ w.r.t. $\langle \succ_1, \succ_2, \succ_3^{\mathrm{mul}} \rangle$)

  obtain $s^{\#} \approx t^{\#} \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \xrightarrow[\mathcal{R}]{\mathrm{top}} \cup \xrightarrow[\mathcal{R}]{>\mathrm{top}} \right) u^{\#} \in \mathrm{minNT}$ ✓ ✓ ✓

qed

Formal proof

theorem

assumes no **minimal** $\mathrm{DP}(\mathcal{R})$- or $\mathcal{R}_{\mathcal{E}}^{\#}$-chain

and $s^{\#} \in \mathrm{minNT}$ **shows** False

proof(induct $\langle s^{\#}, s^{\#}, \nabla s \rangle$ w.r.t. $\langle \succ_1, \succ_2, \succ_3^{\mathrm{mul}} \rangle$)

obtain $s^{\#} \approx t^{\#} \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \cup \xrightarrow[\mathcal{R}]{\mathrm{top}} \cup \xrightarrow[\mathcal{R}]{>\mathrm{top}} \right) u^{\#} \in \mathrm{minNT}$ ✓ ✓ ✓

qed

everything is terminating w.r.t. $\mathcal{R}$ (because of $^{\#}$)!

Usable rules [Hirokawa & Middeldorp '04]

- Allows ignoring "irrelevant" rules in $\mathcal{R}$
- Basic idea:

list all possible rewriting steps, and allow choosing any.

$$f^{\#}(s_1) \xrightarrow[\mathcal{R}]{} f^{\#}(s_2) \xrightarrow[\mathcal{R}]{} \cdots \xrightarrow[\mathcal{R}]{} f^{\#}(s_n)$$

$$\Rightarrow \quad f^{\#}(s_1 \circ s_2 \circ s_3 \circ \cdots) \xrightarrow[\mathcal{C}_e]{}^{*} f^{\#}(s_n)$$

where $\mathcal{C}_e := \begin{cases} x \circ y \to x \\ x \circ y \to y \end{cases}$

$\{ s_1 \circ s_2 \circ s_3 \circ \cdots \}$ must be finite, i.e., terminating w.r.t. $\mathcal{R}$!

Experiments

	AProVE						NaTT			
	no-ACDP		certified		full		certified		full	
	#	time	#	time	#	time	#	time	#	time
yes	82	166.3	128	560.3	128	508.5	78	6.2	113	86.3
no	0	-	0	-	2	21.2	0	-	1	0.5
maybe	63	189.7	14	310.6	12	340.6	67	46.3	31	149.2
timeout	0	-	3	1080.0	3	1080.0	0	-	0	-

Related work

Methods	extra work w.r.t. DP framework	Correctness	Minimality		
				usable rules	subterm criterion
Marché & Urbain '98	flattening	no	no	-	-
Kusakari & Toyama '01	nested #	yes...	yes...	no	no
Giesl & Kapur '01	none	maybe	no	-	-
Alarcón+ '10	none	yes...	yes...	no	no
Ours	none	YES	yes	yes	yes

Conclusion

- Yet another AC-DP framework
  - Formalized
  - Admits "usable rules" and "subterm criterion"
  - Less DPs (as in Kusakari-Toyama)
  - Easy to implement (as in Giesl-Kapur)
- Future Work
  - Implement subterm criterion
  - More general than AC?

AC-DP framework

trivial: $$\frac{}{\langle \emptyset / \mathcal{Q}, \mathcal{R} / \mathcal{E} \rangle}$$

reduction pair: $$\frac{\langle \mathcal{P}' / \mathcal{Q}', \mathcal{R} / \mathcal{E} \rangle}{\langle \mathcal{P} / \mathcal{Q}, \mathcal{R} / \mathcal{E} \rangle}$$ if $\mathcal{P}' = \mathcal{P} \setminus \succ$ and $\mathcal{Q}' = \mathcal{Q} \setminus \succ$ and $\mathcal{P} \cup \mathcal{Q} \cup \mathcal{R} \cup \mathcal{E} \subseteq \succcurlyeq$

dependency graph: $$\frac{\langle \mathcal{P}_1 / \mathcal{Q}_1, \mathcal{R} / \mathcal{E} \rangle \quad \langle \mathcal{P}_2 / \mathcal{Q}_2, \mathcal{R} / \mathcal{E} \rangle \quad \cdots \quad \langle \mathcal{P}_n / \mathcal{Q}_n, \mathcal{R} / \mathcal{E} \rangle}{\langle \mathcal{P} / \mathcal{Q}, \mathcal{R} / \mathcal{E} \rangle}$$ if ...

init: $$\frac{\langle \mathrm{DP}(\mathcal{R}) / \mathbf{DP}(\mathcal{E}), \mathcal{R} / \mathcal{E} \rangle \qquad \langle \mathcal{R}^{\#}_{\mathcal{E}} / \mathbf{DP}(\mathcal{E}), \mathcal{R} / \mathcal{E} \rangle}{\mathcal{R} / \mathcal{E} \text{ is terminating}}$$