The Class of μ-Continuous Chomsky Algebras is Closed under Matrix Rings

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Chomsky Algebras

A Chomsky algebra $(M, +, \cdot, 0, 1)$ is an idempotent semiring where every finite system of polynomial inequations

$$\begin{array}{rcl} x_1 & \geq & p_1(x_1, \ldots, x_n, y_1, \ldots, y_m), \\ & \vdots & \\ x_n & \geq & p_n(x_1, \ldots, x_n, y_1, \ldots, y_m), \end{array} \quad \text{or} \quad \bar{x} \geq \bar{p}(\bar{x}, \bar{y}), \tag{1}$$

has least solutions, i.e. for all $\bar{b} \in M^m$ there is a (unique) least $\bar{a} = a_1, \ldots, a_n \in M^n$ such that

$$a_i \geq p_i^M(\bar{a}, \bar{b}) \text{ for } i = 1, \ldots, n.$$

Here $\leq$ is the natural partial order on M defined by

$$a \leq b \text{ iff } a + b = b.$$

The system $\bar{x} \geq \bar{p}(\bar{x}, \bar{y})$ is a context-free grammar with nonterminals $\bar{x} = x_1, \ldots, x_n$ and terminals $\bar{y} = y_1, \ldots, y_m$.

Example

Let $(X^*, \cdot, \epsilon)$ be the monoid of all finite words of elements of X. Its power set is an idempotent semiring:

$$(\mathcal{P}X^*, +, \cdot, 0, 1), \text{ with } \begin{array}{rcl} 0 & := & \emptyset, \quad 1 \ := \ \{\epsilon\}, \\ A + B & := & A \cup B, \\ A \cdot B & := & \{a \cdot b \mid a \in A, b \in B\}. \end{array}$$

For a vector $\bar{B}$ of m languages, $\bar{x} \geq \bar{p}(\bar{x}, \bar{y})$ leads to an increasing sequence $\bar{A}_k = (A_{k,1}, \ldots, A_{k,n})$ of language vectors by

$$A_{0,i} := \emptyset, \quad A_{k+1,i} := p_i^{\mathcal{P}X^*}(\bar{A}_k, \bar{B}), \quad i = 1, \ldots, n.$$

The least solution of the inequation system, relative to $\bar{B}$, is

$$\bar{A} := \bigcup \{\bar{A}_k \mid k \in \mathbb{N}\}.$$

Therefore, $(\mathcal{P}X^*, +, \cdot, 0, 1)$ is a Chomsky algebra.

The inclusion $\bar{A} \supseteq \bar{p}^{\mathcal{P}X^*}(\bar{A}, \bar{B})$ follows from the fact that $+$ and $\cdot$ are compatible with arbitrary unions (sup-continuous):

for all $A, B \subseteq X^*$ and $\emptyset \neq \mathcal{C} \subseteq \mathcal{P}X^*$,

$$\begin{aligned} A + \bigcup \mathcal{C} &= \bigcup \{A + C \mid C \in \mathcal{C}\}, \\ A \cdot \bigcup \mathcal{C} \cdot B &= \bigcup \{A \cdot C \cdot B \mid C \in \mathcal{C}\}. \end{aligned}$$

Example

$Rel\,(X) := (\mathcal{P}(X \times X), +, \cdot, 0, 1)$, the set of all binary relations on X with union as $+$, relation composition ; as $\cdot$, the empty relation as 0 and the identity relation as 1.

Example

The set $\mathcal{C}X^*$ of context-free languages over X is the smallest $\mathcal{L} \subseteq \mathcal{P}X^*$ such that

(i) each finite subset of $X \cup \{\epsilon\}$ is in $\mathcal{L}$, and

(ii) if $\bar{x} \geq \bar{p}(\bar{x}, \bar{y})$ is a polynomial system and $\bar{B} \in \mathcal{L}^m$, then the least $\bar{A} \in (\mathcal{P}X^*)^n$ with $\bar{A} \supseteq \bar{p}^{\mathcal{P}X^*}(\bar{A}, \bar{B})$ belongs to $\mathcal{L}^n$.

With the operations inherited from $\mathcal{P}X^*$, $(\mathcal{C}X^*, +, \cdot, 0, 1)$ is a Chomsky algebra. Likewise:

- the set $\mathcal{C}(X^2) \subseteq \mathcal{P}(X^2)$ of context-free relations on the set X,
- the set $\mathcal{C}M \subseteq \mathcal{P}M$ of context-free subsets of a monoid M.

The regular languages over X do not form a Chomsky algebra, as they don't have solutions for inequations like $axb + 1 \leq x$.

Goals of this talk: For any Chomsky algebra M:

- The matrix ring $Mat_{n,n}(M)$ is also a Chomsky algebra.
- If least solutions of systems $\bar{x} \geq \bar{t}$ can be computed in M iteratively, they can so be computed in $Mat_{n,n}(M)$.

Chomsky algebras were introduced by Grathwohl, Henglein, Kozen 2013 in providing an infinitary complete axiom system for the equational theory of context-free grammars as fixed-point terms.

Theorem (Grathwohl,Henglein,Kozen FICS 2013)

The axioms of idempotent semirings and μ-continuity are sound and complete for the equational theory of the context-free languages.

μ-terms and Park μ-semirings

Let X be an infinite set of variables. The μ-terms over X are defined by

$$t := x \mid 0 \mid 1 \mid (s \cdot t) \mid (s + t) \mid \mu x t.$$

A term t is is algebraic or a polynomial if it does not contain μ .

- $free(t)$ is the set of variables having a free occurrence in t. $t(x_1, \ldots, x_n)$ indicates $free(t) \subseteq \{x_1, \ldots, x_n\}$.
- $t[x/s]$ is the result of substituting all free occurrences of x in t by s, renaming bound variables of t to avoid variable capture.
- The μ-depth of a term is: 0 for x, 0, 1; is μ-depth$(t) + 1$ for $\mu x t$; and is $\max\{\mu\text{-depth}(s), \mu\text{-depth}(t)\}$ for $(s + t)$, $(s \cdot t)$.

Note: We'll write $\mu x.t[y/s]$ for $\mu x(t[y/s])$, using . to save the brackets of the metalanguage. (We prefer $\mu x(t + s)$ over $\mu x.t + s$.)

A partially ordered μ-semiring $(M, +, \cdot, 0, 1, \leq)$ is a semiring $(M, +, \cdot, 0, 1)$ with a partial order $\leq$ on M, where every μ-term t defines a function $t^M : (X \to M) \to M$, so that for all variables $x \in X$, terms s, t and valuations $g, h : X \to M$ we have:

1. $$\begin{array}{rcllrcl} 0^M(g) & = & 0, & & (s+t)^M(g) & = & s^M(g) + t^M(g), \\ 1^M(g) & = & 1, & & (s \cdot t)^M(g) & = & s^M(g) \cdot g^M(g), \\ x^M(g) & = & g(x), & (\mu\text{-rule}) \text{ if } s^M \leq t^M, \text{then } \mu x s^M \leq \mu x t^M, \end{array}$$
2. (monotonicity) if $g \leq h$ pointwise, then $t^M(g) \leq t^M(h)$,
3. (coincidence) if g, h agree on $\mathit{free}(t)$, then $t^M(g) = t^M(h)$,
4. (substitution) $t[x/s]^M(g) = t^M(g[x/s^M(g)])$.

For $t^M(g)$ we also write $t^M[x_1/a_1, \ldots, x_n/a_n]$ or $t^M(a_1, \ldots, a_n)$.

A first-order formula built from (in)equations holds in M if it is true for every valuation $g : X \to M$.

A Park μ-semiring is a partially ordered μ-semiring M where for all terms t and variables x, y, the following "Park axioms" hold in M:

$$
\begin{aligned}
t[x/\mu x t] \quad &\leq \quad \mu x t, &(2)\\
t[x/y] \leq y \quad &\rightarrow \quad \mu x t \leq y. &(3)
\end{aligned}
$$

Then the following also hold in M:

$$
\begin{aligned}
t[x/\mu x t] \quad &= \quad \mu x t,\\
\mu y.t[x/y] \quad &= \quad \mu x t, \quad \text{for } y \notin \mathit{free}(t).
\end{aligned}
$$

Park's axioms imply that $\mu x t^M(g)$ is the least solution of $t \leq x$ in M, g, i.e. the least $a \in M$ such that $t^M(g[x/a]) \leq a$.

Lemma (c.f. Grathwohl/Henglein/Kozen 2013)

Every Chomsky algebra M is an idempotent, partially ordered μ-semiring, if for all terms t, variables x and valuations $g : X \to M$

$$\mu x t^M(g) := \text{the least } a \in M \text{ such that } t^M(g[x/a]) \leq a. \tag{4}$$

Moreover, every inequation system $\bar{t}(\bar{x}, \bar{y}) \leq \bar{x}$ with μ-terms $\bar{t}(\bar{x}, \bar{y})$ has least solutions in M, i.e. for all parameters $\bar{b}$ from M there is a least tuple $\bar{a}$ in M such that $\bar{t}^M(\bar{a}, \bar{b}) \leq \bar{a}$.

Proof: Simultaneously by induction on the μ-depth of terms.

Corollary

Every Chomsky algebra M, in particular $\mathcal{C}X^*$, is a Park μ-semiring. For every μ-term and $g : X \to \mathcal{C}X^*$, $t^{\mathcal{C}X^*}(g) = t^{\mathcal{P}X^*}(g)$.

Vector versions and matrix ring of Park μ-semirings

To name the least solution of a system $\bar{x} \geq \bar{t}$ of n inequations, one might introduce terms $\mu\bar{x}\bar{t}$ using an n-ary fixed-point operator μ. As is well-known, a unary μ normally suffices:

Theorem (Bekić 1984)

Let $(M, \leq)$ be a partially ordered set in which every countable increasing chain $A = \{a_i \mid i \in \mathbb{N}\}$ has a least upper bound, $\sum A$. Suppose $f, g : M^2 \to M$ are continuous in each component, i.e. $f(\sum A, b) = \sum\{f(a, b) \mid a \in A\}$ for countable chains A, etc. Then the least solution of the system $(x, y) \geq (f(x, y), g(x, y))$ can be obtained from least solutions of single inequations:

$$\begin{aligned} &\mu(x, y)(f(x, y), g(x, y)) \\ &\quad = \ (\mu x.f(x, \mu y.g(x, y)), \mu y.g(\mu x.f(x, y), y). \end{aligned} \tag{5}$$

For an n-dimensional inequation system $\bar{x} \geq \bar{t}$, we define an n-tuple $\mu\bar{x}\bar{t}$ of μ-terms by recursively using Bekić's equations (5).

- If $n = 1$, then $\mu\bar{x}\bar{t} := \mu x_1 t_1$.
- If $n > 1$, $\bar{x} = (\bar{y}, \bar{z})$ and $\bar{t} = (\bar{r}, \bar{s})$ with term vectors $\bar{r}, \bar{s}$ of lengths $|\bar{y}|, |\bar{z}| < n$, then $\mu\bar{x}\bar{t}$ is[1]

$$\mu(\bar{y}, \bar{z})(\bar{r}, \bar{s}) := (\mu\bar{y}.\bar{r}[\bar{z}/\mu\bar{z}\bar{s}], \mu\bar{z}.\bar{s}[\bar{y}/\mu\bar{y}\bar{r}]). \tag{6}$$

However, Bekić's theorem does *not* imply that $\mu\bar{x}\bar{t}$ denotes the least solution of $\bar{x} \geq \bar{t}$ in Chomsky algebras, like $\mathcal{C}X^*$, as these need not be closed under unions of countable increasing chains.

To see that $\mu\bar{x}\bar{t}$ denotes the least solution of $\bar{x} \geq \bar{t}$ in a Chomsky algebra M, we show that Park's axioms for term vectors hold in M.

[1] Recall that $\mu x t[y/s]$ differs from $\mu x.t[y/s] := \mu x(t[y/s])$.

For term vectors $\bar{s}$, $\bar{t}$ of the same dimension, let $\bar{s} = \bar{t}$ resp. $\bar{s} \leq \bar{t}$ be the conjunction of all $s_i = t_i$ resp. $s_i \leq t_i$. For $\bar{t} = (t_1, \ldots, t_n)$ we write $\bar{t}^M(g)$ for $(t_1^M(g), \ldots, t_n^M(g))$.

Lemma

Let M be a Park μ-semiring. For all vectors $\bar{t}$ of terms and $\bar{x}, \bar{y}$ of variables, of the same dimension, the vector versions of (2) and (3),

$$\bar{t}[\bar{x}/\mu\bar{x}\bar{t}] \quad \leq \quad \mu\bar{x}\bar{t}, \tag{7}$$

$$\bar{t}[\bar{x}/\bar{y}] \leq \bar{y} \quad \to \quad \mu\bar{x}\bar{t} \leq \bar{y}, \tag{8}$$

hold in M. Moreover, for any $g : X \to M$,

$$\mu\bar{x}\bar{t}[\bar{y}/\bar{s}]^M(g) = \mu\bar{x}\bar{t}^M(g[\bar{y}/\bar{s}^M(g)]), \tag{9}$$

if no variable of $\bar{x}$ is free in the terms $\bar{s}$.

Proof: induction on dimension, simultaneously for (7), (8) and (9).

Corollary

For any Chomsky algebra M and valuation $g : X \to M$, $\mu\bar{x}\bar{t}^M(g)$ is the least $\bar{a}$ such that $\bar{t}^M(g[\bar{x}/\bar{a}]) \leq \bar{a}$.

Corollary

If M is a Park μ-semiring, the vector version of the μ-rule holds: for vectors $\bar{s}$, $\bar{t}$ of terms and $\bar{x}$ of different variables, all of the same dimension, if $\bar{s}^M \leq \bar{t}^M$, then $\mu\bar{x}\bar{s}^M \leq \mu\bar{x}\bar{t}^M$.

Theorem (Ésik/L. 2004)

If M is a Park μ-semiring, so is $N := Mat_{n,n}(M)$.

Proof: With matrix operations, $(N, +, \cdot, 0, 1)$ is a semiring, as M is. To define the term functions $t^N : (X \to N) \to N$, for each variable $x \in X$ we fix n^2 fresh distinct variables $x_{i,j}$, $1 \leq i, j \leq n$. For each term t, define a vector t' of n^2 terms recursively by

$$\begin{array}{rcl}
x' & := & (x_{i,j}), \\
0' & := & 0_{n,n}, \\
1' & := & 1_{n,n},
\end{array}
\qquad
\begin{array}{rcl}
(s+t)' & := & s' + t', \\
(s \cdot t)' & := & s' \cdot t', \\
(\mu x t)' & := & \mu x' t',
\end{array}$$

using the usual matrix operations for matrices of terms, and $\mu x' t'$ is the term vector $\mu \bar{x} \bar{t}$ defined by Bekić for the inequations $x' \geq t'$.

Any valuation $g : X \to N$ is obtained from a valuation $\hat{g} : X \to M$ by $g(x) = (a_{i,j})$, where $a_{i,j} = \hat{g}(x_{i,j})$ for $1 \leq i, j \leq n$.

Define the term function t^N by

$$t^N(g) := ({t'_{i,j}}^M(\hat{g})), \qquad \text{where } t'_{i,j} \text{ is the } (i,j)\text{-th entry of } t'. \quad (10)$$

N is a partially ordered μ-semiring:

1. μ-rule: since the vector version of the μ-rule holds in M.

2.,3. monotonicity and coincidence for term function t^N: from those of the of the term functions ${t'}^M_{i,j}$.

4. substitution: since its vector version holds in M.

N is a Park μ-semiring:

5. the vector versions of Park's axioms hold in M.

Theorem

If M is a Chomsky algebra, so is $N := Mat_{n,n}(M)$.

Proof We saw that M is an idempotent Park μ-semiring, whose matrix ring N also is and so satisfies the vector versions of Park's axioms. In particular, every system $\bar{x} \geq \bar{p}(\bar{x}, \bar{y})$ of polynomial inequations has least solutions in N. So, N is a Chomsky algebra.

In Park μ-semirings, we know that $\mu\bar{x}\bar{t}$ denotes the minimal solution of $\bar{x} \geq \bar{t}$. We don't know whether minimal solutions can be iteratively approximated.

μ-Continuity

The usual way to compute the simultaneous least fixed point $\bar{A}$ of $\bar{x} \geq \bar{t}(\bar{x}, \bar{y})$ in $\mathcal{P}X^*$ relative to $\bar{B}$ is to approximate it by its finite stages $\bar{A}_m$ and (componentwise) take their union, i.e.

$$\bar{A} := \bigcup_{m \in \mathbb{N}} \bar{A}_m, \quad \text{where} \quad \bar{A}_0 := \bar{\emptyset}, \quad \bar{A}_{m+1} := \bar{t}^{\,\mathcal{P}X^*}(\bar{A}_m, \bar{B}).$$

The continuity of $+$ and $\cdot$ in $\mathcal{P}X^*$ imply that $\bar{A} = \mu\bar{x}\bar{t}^{\,\mathcal{P}X^*}(\bar{B})$.

Lemma

In $M = \mathcal{P}X^*$ or $M = \mathcal{C}X^*$, all term functions are continuous: for a term t, valuation $g : X \to M$ and increasing chain $\{A_i \mid i \in \mathbb{N}\}$ of elements whose union is in M,

$$t^M(g[y/\bigcup_{i \in \mathbb{N}} A_i]) = \bigcup_{i \in \mathbb{N}} t^M(g[y/A_i]). \tag{11}$$

A Chomsky algebra M is μ-continuous, if for all μ-terms $t(x, \bar{y})$ and all $a, b, \bar{c} \in M$, the $\sum$ on the rhs exists and

$$a \cdot \mu x t^M[\bar{y}/\bar{c}] \cdot b = \sum \{a \cdot (mxt)^M[\bar{y}/\bar{c}] \cdot b \mid m \in \mathbb{N}\}, \qquad (12)$$

where the term mxt, the m-fold iteration of t in x, is defined by

$$0xt := 0, \ (m+1)xt := t[x/mxt].$$

(A Kleene algebra is *-continuous if $ac^*b = \sum\{ac^m b \mid m \in \mathbb{N}\}$.)

Theorem (Grathwohl,Henglein,Kozen 2013)

The Chomsky algebra $\mathcal{C}X^*$ of all context-free languages over X is μ-continuous. Hence, for all terms t and valuations $g : X \to \mathcal{C}X^*$,

$$\mu x t^{\mathcal{C}X^*}(g) = \bigcup \{mxt^{\mathcal{C}X^*}(g) \mid m \in \mathbb{N}\}.$$

The iterative computation of least fixed points is used in [2] to interpret μ-terms in the semiring $\mathcal{C}X^*$:

The canonical interpretation L of μ-terms over X in $\mathcal{C}X^*$ is

$$
\begin{array}{rcl@{\qquad}rcl}
L(x) & = & \{x\} & L(s+t) & = & L(s) \cup L(t) \\
L(0) & = & \emptyset & L(s \cdot t) & = & \{uv \mid u \in L(s), v \in L(t)\} \\
L(1) & = & \{\epsilon\} & L(\mu x t) & = & \bigcup\{L(mxt) \mid m \in \mathbb{N}\}.
\end{array}
$$

By the previous theorem, $L(t) = t^{\mathcal{C}X^*}(L)$ for all terms t.

Remark Grathwohl e.a. prove that an idempotent semiring M with an interpretation of μ-terms satisfying μ-continuity also satisfies Park's axioms.

The next lemma says that with L, for any system $\bar{x} \geq \bar{t}$, the value of $\mu\bar{x}\bar{t}$ is the union of the values of the iterations $m\bar{x}\bar{t}$, where

$$0\bar{x}\bar{t} := \bar{0}, \qquad (m+1)\bar{x}\bar{t} := \bar{t}[\bar{x}/m\bar{x}\bar{t}].$$

Lemma

For all vectors $\bar{x}, \bar{y}, \bar{z}$ of pairwise distinct variables and vectors $\bar{t}, \bar{r}, \bar{s}$ of μ-terms with $|\bar{x}| = |\bar{t}|$, $|\bar{y}| = |\bar{r}|$ and $|\bar{z}| = |\bar{s}|$, we have:

1. $L(\mu\bar{x}\bar{t}) = \bigcup\{L(m\bar{x}\bar{t}) \mid m \in \mathbb{N}\}$,
2. $L(\bar{s}[\bar{x}/\mu\bar{x}\bar{t}]) = \bigcup\{L(\bar{s}[\bar{x}/m\bar{x}\bar{t}]) \mid m \in \mathbb{N}\}$,
3. $L(\mu\bar{z}.\bar{s}[\bar{y}/\mu\bar{y}\bar{r}]) = \bigcup\{L(m\bar{z}.\bar{s}[\bar{y}/k\bar{y}\bar{r}]) \mid m, k \in \mathbb{N}\}$.

Proof (lengthy): by simultaneous induction on the vector length of $|\bar{x}| = |\bar{y}| + |\bar{z}|$ with $|\bar{y}|, |\bar{z}| < |\bar{x}|$.

The proof uses the vector version of the substitution property, and the definition of L in 1. for $|\bar{t}| = 1$.

The following lemma will help us to transfer the vector version of the μ-continuity condition from L resp. $\mathcal{C}X^*$ to an arbitrary μ-continuous Chomsky algebra.

Lemma (Grathwohl,Henglein,Kozen 2013)

Let M be a μ-continuous Chomsky algebra and $g : X \to M$. Then, for all μ-terms s, t, u,

$$(stu)^M(g) = \sum\{(swu)^M(g) \mid w \in L(t)\},$$

where, for $x_1 \cdots x_k \in X^*$, $(x_1 \cdots x_k)^M(g) := g(x_1) \cdot^M \ldots \cdot^M g(x_k)$.

Corollary

For any vectors $\bar{s}, \bar{t}, \bar{u}$ of μ-terms of equal length, any μ-continuous Chomsky-algebra M and valuation $g : X \to M$,

$$(\bar{s} \cdot \bar{t} \cdot \bar{u})^M(g) = \sum\{(\bar{s} \cdot \bar{w} \cdot \bar{u})^M(g) \mid \bar{w} \in L(\bar{t})\}. \qquad (13)$$

Applied to $L(\mu\bar{x}\bar{t})$ this gives the vector version of μ-continuity:

Corollary

Let M be a μ-continuous Chomsky algebra and $g : X \to M$. Then

$$\bar{a} \cdot \mu\bar{x}\bar{t}^M(g) \cdot \bar{b} = \sum\{\bar{a} \cdot m\bar{x}\bar{t}^M(g) \cdot \bar{b} \mid m \in \mathbb{N}\},$$

for any term vector $\bar{t}$ and vectors $\bar{a}, \bar{b}$ of elements of M of the same length as $\bar{t}$.

The matrix ring of a μ-continuous Chomsky algebra

Theorem

If M is a μ-continuous Chomsky algebra, so is $N := Mat_{n,n}(M)$.

Proof. We saw that N is a Chomsky algebra, since M is.

Suppose M is μ-continuous and $n > 1$. Let $A, B \in N$, $t(x, \bar{y})$ a μ-term and $g : X \to N$. We have to show

$$A \cdot \mu x t^N(g) \cdot B = \sum \{A \cdot m x t^N(g) \cdot B \mid m \in \mathbb{N}\}. \qquad (14)$$

By definition, $\mu x t^N(g) = ((\mu x t)'^{M}_{i,j}(\hat{g}))$, where $(\mu x t)' = \mu x' t'$ is obtained from the matrices

$$x' = (x_{i,j}) \quad \text{and} \quad t' = (t_{i,j})$$

of new variables $x_{i,j}$ and μ-terms $t_{i,j}(x', \bar{y}')$ using Bekić's definition for the n^2 inequations $t' \leq x'$. Accordingly, we have a square term matrix $(m x t)' = m x' t'$ for the m-th iteration of t' in x'.

Concerning μ-continuity of matrix multiplication, consider the (i,j)-th entry and use (*) the vector version of μ-continuity in M:

$$
\begin{aligned}
& (A \cdot \mu x t^N(g) \cdot B)_{i,j} \\
&= \sum_{k,l \le n} a_{i,k} \cdot^M (\mu x t^N(g))_{k,l} \cdot^M b_{l,j} \\
&= \sum_{k,l \le n} a_{i,k} \cdot^M (\mu x t)'^M_{k,l}(\hat{g}) \cdot^M b_{l,j} \\
&= \sum_{k,l \le n} \sum \{ a_{i,k} \cdot^M (m x t)'^M_{k,l}(\hat{g}) \cdot^M b_{l,j} \mid m \in \mathbb{N} \} \quad (*) \\
&= \sum \{ \sum_{k,l \le n} a_{i,k} \cdot^M (m x t)'^M_{k,l}(\hat{g}) \cdot^M b_{l,j} \mid m \in \mathbb{N} \} \\
&= \sum \{ \sum_{k,l \le n} a_{i,k} \cdot^M ((m x t)^N(g))_{k,l} \cdot^M b_{l,j} \mid m \in \mathbb{N} \} \\
&= \sum \{ (A \cdot m x t^N(g) \cdot B)_{i,j} \mid m \in \mathbb{N} \}
\end{aligned}
$$

Questions

- Can the result be proven by induction on the matrix dimension (instead of the vector dimension)?
  If so, is the Kleene algebra case an instance of the proof?
  Recall that for matrices A over a Kleene algebra M

  $$\mu X(AX + 1) = A^*,$$

  and there is a recursion formula to compute A^* by induction on its dimension.
- For an upper triangular Boolean matrix A, Valiant's algorithm computes the reflexive transitive closure

  $$A^* = \mu X(A + 1 + XX)$$

  efficiently. Can it be extended to arbitrary Boolean matrices?

- M.Hopkins (2008) has an algebraic theory which defines, for any monoid M, a Chomsky hierarchy

$$\mathcal{R}M \subset \mathcal{C}M \subset \mathcal{T}M \subset \mathcal{P}M$$

  of the regular, context-free, turing (r.e.) and arbitrary subsets, each of which form an idempotent semiring.
  - Is $\mathcal{C}M \subseteq \mathcal{P}M$ as definied above the same as Hopkins' $\mathcal{C}M$?
  - Is $\mathcal{T}$ closed under matrix ring formation?
  - Is $\mathcal{C}M$ the ideal closure of M for suitabe "μ-ideals" $I \subseteq M$?
- If we adjoin to an idempotent semiring M a solution to a polynomial inequation $x \geq p(x)$, which other inequations $x \geq q(x)$ get solvable?

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